GENERALIZATION OF RHODES THEOREM TO A SEQUENCE OF SELF MAPS

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ABSTRACT *In this research article we considered a seq- of self maps on a complete 2-metric space satisfying contractive type condition. The Rhodes fi- point theorem can be obtained as a corollary to this generalization*

AMS Subject Classification 47HI0, 54H25

Key words : Fi -point, 2- Metric space , Caucy- sequence, Self mapping

INTRODUCTION

In 1906, *Frechet* ([1] & [2]) introduced the notion of metric(met) as an abstract generalization of length concept. The notion of 2-metric was introduced by *Gahler* [3] in 1963 as an abstract generalization of the concept of area function for Euclidean triangles. The concept of 2-metric attracted the attention of many researchers in the field of fi- point theory. Many authors like *Iseki* [4], *Khan* [5], *Rhoades* [6], *Lal* and *Singh* [6] etc. probed deeply into this area and established several fi-point theorems in 2-met space setting as generalizations or extensions to the metric fi- point theorems. Several fi- point theorems appeared in 2-met space setting analogous to the fi- point theorems in met spaces.

1. PRELIMINARIES

In this section we define Fi-point , 2- Met space and Caucy sequence

Definition-1.1 Let X be a non-empty set. A function F from X into itself is called a self map on X. A point x belongs to X is called a fi-point of a self map if F(x) = x

Definition-1.2: A *2-metr.* on a non-empty set Ω is a function $\lambda : \Delta \to ¡$, satisfying the following properties.

(a) $\lambda(u,v,w) = 0$ if at least two of u,v,w are equal

(b) for each pair of distinct points u,v in Ω there exists a point $w \in \Omega$ such that $\lambda(u,v,w) \neq 0$

(c) $\lambda(u,v,w) = \lambda(u,w,v) = \lambda(v,w,u) \quad \forall \; u,v \text{ and } w \text{ in } \Omega$

(d) $\lambda(u,v,w) \leq \lambda(u,v,a) + \lambda(u,a,w) + \lambda(a,v,w) \quad \forall \; u,v,w \text{ and } a \text{ in } \Omega$

then λ is called a *2-metric* on Ω and the pair (Ω, λ) is called a *2-metric space.*

Example-1.3: Consider the real line R. Define $d : \mathbb{R}^3 \to \mathbb{R}$ by

$$d(x,y,z) = \min\{|x-y|,|y-z|,|z-x|\}.$$

Then d forms a 2-met on R.

Defination 1.4: A sequence $\{x_n\}$ in a 2-normed linear space $(L,\|.,.\|)$ is said to be a *Caucy sequence* if there exist two points $y,z \in L$ such that

(a) y and z are linearly independent

(b) $\lim\limits_{m,n\to\infty}\|x_n - x_m, y\| = 0$

(c) $\lim\limits_{m,n\to\infty}\|x_n - x_m, z\| = 0$

1. **A Generalisation of Rhoades theorem**

This section mainly focuses on the generalisation of the Rhoades fi- point theorem which was published by *B.E. Rhoades* in his classical research article [6] of 1979. His fi-point theorem can easily be obtained from the following generalised fi- point theorem as a corollary- 2.3.

Theorem-2.1: Let $m \geq 1$ and $\delta \in (0,1)$. Suppose that $\langle \varphi_n \rangle$ where $n \in ¢_*$, is a sequence of self maps on a complete 2-metric space (Ω, λ) such that

$$\lambda\left(\varphi_0^m(u), \varphi_n^m(v), a\right) \leq \delta \max\Big\{\lambda(u,v,a), \lambda\left(u, \varphi_0^m(u), a\right), \lambda\left(v, \varphi_n^m(v), a\right),$$
$$\frac{1}{2}\left[\lambda\left(u, \varphi_n^m(v), a\right) + \lambda\left(v, \varphi_0^m(u), a\right)\right]\Big\}$$

For every u, v *and* a *in* Ω. Then there exists a uniq common fi- point for each φ_n in Ω .

Proof: Fix a point u_0 in Ω and define a sequence $\langle u_n \rangle$ in Ω such that

$u_{2n-1} = \varphi_0^m(u_{2n-2})$ & $u_{2n} = \varphi_n^m(u_{2n-1})$ where $n \in \mathbb{N}$.

Now we have

$$\lambda(u_{2n}, u_{2n+1}, u_{2n+2}) = \lambda(u_{2n+1}, u_{2n+2}, u_{2n})$$
$$= \lambda\left(\varphi_0^m(u_{2n}), \varphi_n^m(u_{2n+1}), u_{2n}\right)$$
$$\leq \delta \max\Big\{\lambda(u_{2n}, u_{2n+1}, u_{2n}), \lambda\left(u_{2n}, \varphi_0^m(u_{2n}), u_{2n}\right), \lambda\left(u_{2n+1}, \varphi_n^m(u_{2n+1}), u_{2n}\right),$$
$$\frac{1}{2}\left[\lambda\left(u_{2n}, \varphi_n^m(u_{2n+1}), u_{2n}\right) + \lambda\left(u_{2n+1}, \varphi_0^m(u_{2n}), u_{2n}\right)\right]\Big\}$$
$$= \delta\lambda\left(u_{2n+1}, \varphi_n^m(u_{2n+1}), u_{2n}\right)$$
$$= \delta\lambda(u_{2n+1}, u_{2n+2}, u_{2n})$$

$$\Rightarrow \lambda(u_{2n}, u_{2n+1}, u_{2n+2}) \leq \delta\lambda(u_{2n}, u_{2n+1}, u_{2n+2})$$

By Result-1.4.2, it follows that $\lambda(u_{2n}, u_{2n+1}, u_{2n+2}) = 0$.

Now $\lambda(u_1,u_2,a)=\lambda(\varphi_0^m(u_0),\varphi_1^m(u_1),a)$

$\leq \delta \max\{\lambda(u_0,u_1,a),\lambda(u_0,\varphi_0^m(u_0),a),\lambda(u_1,\varphi_1^m(u_1),a)$

$\frac{1}{2}[\lambda(u_0,\varphi_1^m(u_1),a)+\lambda(u_1,\varphi_0^m(u_0),a)]\}$

$= \delta \max\{\lambda(u_0,u_1,a),\lambda(u_1,u_2,a),\frac{1}{2}\lambda(u_0,u_2,a)\}$

$\leq \delta \max\{\lambda(u_0,u_1,a),\lambda(u_1,u_2,a),\frac{1}{2}[\lambda(u_1,u_2,a)+\lambda(u_0,u_1,a)]\}$

Put $K=\max\{\lambda(u_0,u_1,a),\lambda(u_1,u_2,a),\frac{1}{2}[\lambda(u_1,u_2,a)+\lambda(u_0,u_1,a)]\}$

If $K=\lambda(u_1,u_2,a)$ then we obtain $\lambda(u_1,u_2,a)\leq \delta\lambda(u_1,u_2,a)<\lambda(u_1,u_2,a)$

This is a contradiction. So $K\neq \lambda(u_1,u_2,a)$.

If $K=\frac{1}{2}[\lambda(u_0,u_1,a)+\lambda(u_1,u_2,a)]$

then $\lambda(u_1,u_2,a)\leq \frac{\delta}{2}[\lambda(u_0,u_1,a)+\lambda(u_1,u_2,a)]$ → (1)

Since $K=\max\{\lambda(u_0,u_1,a),\lambda(u_1,u_2,a),\frac{1}{2}[\lambda(u_1,u_2,a)+\lambda(u_0,u_1,a)]\}$,

we have $K\geq \lambda(u_0,u_1,a)$

Since we supposed that, we get $K=\frac{1}{2}[\lambda(u_1,u_2,a)+\lambda(u_0,u_1,a)]$

$\frac{1}{2}[\lambda(u_1,u_2,a)+\lambda(u_0,u_1,a)]\geq \lambda(u_0,u_1,a)$

$\Rightarrow \lambda(u_1,u_2,a)+\lambda(u_0,u_1,a)\geq 2\lambda(u_0,u_1,a)$

$\Rightarrow \lambda(u_1,u_2,a)\geq \lambda(u_0,u_1,a)$

But $\lambda(u_1,u_2,a)\leq \frac{\delta}{2}[\lambda(u_0,u_1,a)+\lambda(u_1,u_2,a)]$

So, we get $\lambda(u_0,u_1,a)\leq \frac{\delta}{2}[\lambda(u_0,u_1,a)+\lambda(u_1,u_2,a)]$ → (2)

Adding (1) and (2),

$\lambda(u_0,u_1,a)+\lambda(u_1,u_2,a)\leq \delta[\lambda(u_0,u_1,a)+\lambda(u_1,u_2,a)]$

which is also a contradiction.

So we must have $K=\lambda(u_0,u_1,a)$

Then $\lambda(u_1,u_2,a)\leq \delta\lambda(u_0,u_1,a)$

Similarly $\lambda(u_2,u_3,a)\leq \delta^2\lambda(u_0,u_1,a)$

Continuing in this way we get $\lambda(u_n,u_{n+1},a)\leq \delta^n\lambda(u_0,u_1,a)$

We have

$$\lambda(u_n,u_{n+m},a)\le\lambda(u_n,u_{n+m},u_{n+1})+\lambda(u_n,u_{n+1},a)+\lambda(u_{n+1},u_{n+m},a)$$

$$=\lambda(u_n,u_{n+1},a)+\lambda(u_n,u_{n+m},u_{n+1})+\lambda(u_{n+1},u_{n+m},a)$$

$$\le\delta^n\lambda(u_0,u_1,a)+\lambda(u_n,u_{n+m},u_{n+1})+\lambda(u_{n+1},u_{n+m},u_{n+2})$$

$$+\lambda(u_{n+1},u_{n+2},a)+\lambda(u_{n+2},u_{n+m},a)$$

$$\le\delta^n\lambda(u_0,u_1,a)+\delta^{n+1}\lambda(u_0,u_1,a)+\lambda(u_n,u_{n+m},u_{n+1})$$

$$+\lambda(u_{n+1},u_{n+m},u_{n+2})+\lambda(u_{n+2},u_{n+m},a)$$

$$\le\delta^n\lambda(u_0,u_1,a)+\delta^{n+1}\lambda(u_0,u_1,a)+\delta^{n+2}\lambda(u_0,u_1,a)+\dots$$

$$=\delta^n\left(1+\delta+\delta^2+\dots\right)\lambda(u_0,u_1,a)$$

$$=\delta^n\frac{1}{1-\delta}\lambda(u_0,u_1,a)$$

$$\lambda(u_n,u_{n+m},a)\le\frac{\delta^n}{1-\delta}\lambda(u_0,u_1,a)$$

Hence $\lambda(u_n,u_{n+m},a)\to 0$ as $n\to\infty$.

$\Rightarrow \langle u_n\rangle$ is a Caucy sequence in Ω. Since Ω is complet, $\langle u_n\rangle$ cges to a point u in Ω.

Now we prove that u is a uniq. common fi- point to each φ_n, $n\in ¢_*$.

Consider

$$\lambda\left(u,\varphi_0^m(u),a\right)\le\lambda\left(u,\varphi_0^m(u),u_{2n}\right)+\lambda(u,u_{2n},a)+\lambda\left(u_{2n},\varphi_0^m(u),a\right)$$

$$=\lambda(u,u_{2n},a)+\lambda\left(u,\varphi_0^m(u),u_{2n}\right)+\lambda\left(\varphi_n^m(u_{2n-1}),\varphi_0^m(u),a\right)$$

$$\le\lambda(u,u_{2n},a)+\lambda\left(u,\varphi_0^m(u),u_{2n}\right)+\delta\max\{\lambda(u,u_{2n-1},a),$$

$$\lambda\left(u,\varphi_0^m(u),a\right),\lambda(u_{2n-1},u_{2n},a)\frac{1}{2}\left[\lambda(u,u_{2n},a)+\lambda\left(u_{2n-1},\varphi_0^m(u),a\right)\right]\}$$

When $n\to\infty$ and $u_n\to u$

$$\lambda\left(u,\varphi_0^m(u),a\right)\le\delta\max\left\{\lambda\left(u,\varphi_0^m(u),a\right)\frac{1}{2}\lambda\left(u,\varphi_0^m(u),a\right)\right\}$$

$$\Rightarrow\ \lambda\left(u,\varphi_0^m(u),a\right)\le\delta\ \lambda\left(u,\varphi_0^m(u),a\right)$$

$$\Rightarrow\ \lambda\left(u,\varphi_0^m(u),a\right)=0\quad\text{for all }\ a\in\Omega$$

$$\Rightarrow\ \varphi_0^m(u)=u$$

$\Rightarrow$ u is a fixed point of φ_0^m.

Also we have

$$\lambda\left(u,\varphi_n^m(u),a\right)=\lambda\left(\varphi_0^m(u),\varphi_n^m(u),a\right)$$
$$\leq \delta\max\{\lambda(u,u,a),\lambda\left(u,\varphi_n^m(u),a\right),\lambda\left(u,\varphi_n^m(u),a\right)$$
$$\frac{1}{2}\left(\lambda\left(u,\varphi_n^m(u),a\right)+\lambda\left(u,\varphi_0^m(u),a\right)\right)\}$$
$$=\delta\max\left\{\lambda\left(u,\varphi_n^m(u),a\right),\frac{1}{2}\lambda\left(u,\varphi_n^m(u),a\right)\right\}$$
$$=\delta\,\lambda\left(u,\varphi_n^m(u),a\right)$$

$\therefore$ $\lambda\left(u,\varphi_n^m(u),a\right)\leq\delta\,\lambda\left(u,\varphi_n^m(u),a\right)$

$\Rightarrow$ $\lambda\left(u,\varphi_n^m(u),a\right)=0$

$\Rightarrow$ $\varphi_n^m(u)=u$

Hence u is a fi- point of φ_n^m.

$\therefore$ u is a common fi-point of φ_0^m and φ_n^m $n\in¥$.

If possible $u\neq\upsilon$ be another fixed point.

Then $\varphi_0^m(\upsilon)=\varphi_n^m(\upsilon)=\upsilon$.

Now $\lambda(u,\upsilon,a)=\lambda\left(\varphi_0^m(u),\varphi_n^m(\upsilon),a\right)$
$$\leq\delta\max\{\lambda(u,\upsilon,a),\lambda\left(u,\varphi_0^m(u),a\right),\lambda\left(\upsilon,\varphi_n^m(\upsilon),a\right),$$
$$\frac{1}{2}\left(\lambda\left(u,\varphi_n^m(\upsilon),a\right)+\lambda\left(\upsilon,\varphi_0^m(u),a\right)\right)\}$$
$$=\delta\max\left\{\lambda(u,\upsilon,a),0,0,\frac{1}{2}\left[\lambda(u,\upsilon,a)+\lambda(u,\upsilon,a)\right]\right\}$$
$$=\delta\,\lambda(u,\upsilon,a)$$

$\therefore$ $\lambda(u,\upsilon,a)\leq\delta\lambda(u,\upsilon,a)$

$\Rightarrow$ $\lambda(u,\upsilon,a)=0$

$\Rightarrow$ $u=\upsilon$

$\therefore$ u is a uniq common fi- point of φ_0^m and φ_n^m.

If we put $m=1$, then z is a uniq common fixed point of φ_n in Ω, where $n\in¢_*$. ■

Results -2.2: The following Corollary-2.3 is proved in [6] by *Rhoades* and the proof follows from Theorem-2.1 by taking $m=1$, $\varphi_0=\varphi$ and $\varphi_n=\psi$ for every $n\in¥$.

Conclusions -2.3 : Let $\delta\in(0,1)$. Let φ and ψ be two self maps on a complete 2-met. space (Ω,λ) satisfying

$$\lambda(\varphi(u),\psi(v),a) \le \delta \max\Big\{\lambda(u,v,a),\lambda(u,\varphi(u),a),\lambda(v,\psi(v),a),\frac{1}{2}\big(\lambda(u,\psi(v),a)+\lambda(v,\varphi(u),a)\big)\Big\}$$

for every u,v,a in Ω. Then φ and ψ have a uniq common fi- point in Ω.

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