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NON-ASSOCIATIVE RINGS AND ALGEBRAS

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ABSTRACT:

Non associative algebras can be applied, either directly or using their particular methods, to many other branches of Mathematics and other Sciences. Here emphasis will be given to two concrete applications of non associative algebras. In the first one, an application to group theory in the line of the Restricted Burnside Problem will be considered. The second one opens a door to some applications of non-associative algebras to Error correcting Codes and Cryptography.

I. INTRODUCTION

By a non-associative algebra is meant a vector space which is equipped with a bilinear multiplication. If the multiplication is associative, we have the familiar notion of an associative algebra. Also Lie algebras are an essential tool for the study of Lie groups, as is well known. However, we shall not be particularly concerned with associative or Lie algebras today, except as models of what well-behaved non associative algebras should be. The study of algebras which are not associative is not a recent development. The 8-dimensional algebra of Cayley numbers was known as early as 1845.2 However, it is within the last fifteen years that the study has received its greatest impetus. With a few notable exceptions—and always excepting Lie algebras of course—there were only isolated results before that time. By now a pattern is emerging, for certain finite-dimensional algebras at least, and this paper is an exposition of some of the principal results achieved recently in the structure and representation of finite-dimensional non associative algebras. A. A. Albert has been the prime mover in this study. The depth and scope of the results, at least for one class of non associative algebras, may be judged by the fact that N. Jacobson will give the Colloquium lectures at the Summer Meeting of the Society on the topic of Jordan algebras. Except for this reference to these two men, I shall not attempt to identify the authors of any theorems in this talk. The bibliography of the published paper will speak for itself. 2. The associative and Lie theories as models. Let F be an arbitrary field and A be a finite-dimensional associative algebra over F. It is well known that there is an ideal N> called the radical of A, which is the unique maximal nilideal of A (that is, the maximal ideal consisting entirely of nilpotent elements). Furthermore, N is nilpotent in the sense that there is an integer / with the property that any product Z1Z2 • • • zt of t elements from N is zero; hence N is also the unique maximal nilpotent ideal of A. Modulo this radical the algebra is semi simple; that is, the difference algebra A/N has radical equal to zero. Moreover, any semi simple associative algebra is uniquely expressible as a direct sum 5i© • • • © Sr of simple two-sided ideals (where an algebra is simple provided it has no proper ideals and is not a 1-dimensional algebra in which all products are zero). Any simple associative algebra 5 is the Kronecker product FSXD (over F) of the total matric algebra Fs of dimension s 2 and a division algebra D over F, where 5 is unique and D is uniquely determined up to isomorphism. Hence (up to a determination of all division algebras D over F) the structure of any semi simple associative algebra over F is known. Let A/N be separable (that is, the center of each simple component is a separable field over F\ this would always be the case if F were of characteristic 0). Then A has a Wedderburn decomposition A =S-\-N where 5 is a sub algebra of A isomorphic to A/N and S+N is a vector space direct sum. This decomposition is unique up to an inner automorphism of A in the following strong sense. Let z be any nilpotent element in A. Then, even though there is no identity element 1 in Ay the meaning of (1—z)a(l — z)"~l is clear, and the mapping Gz \ a—>(1— z)a{\ — z)"1 is an inner automorphism of A. Suppose that A has Wedderburn decompositions A =S+N=Si + N. Then there is an element z of the radical N such that the corresponding inner automorphism Gz maps 5i onto S (and of course leaves N invariant). This of course has been but the briefest of sketches of the associative structure theory, and omits many important features. It is astonishing, however, how closely the structure of Lie algebras of characteristic 0 parallels the associative theory up to this point. Let F be a field of characteristic 0 and A be a finite-dimensional Lie algebra over F. Then the radical N of A is not the maximal nilideal (since A itself is a nilalgebra, the square of every element being zero), nor is it in general the maximal nilpotent ideal of A. It is an ideal between these two. Define B™=B,]3 = (£)2 . Then B is solvable in case there is an integer k such that J3(fc)=0, and the radical N of A is the unique maximal solvable ideal of A. With this definition of radical, the difference algebra A/N is semi simple and is uniquely expressible as a direct sum of simple two-sided ideals. If F is algebraically closed, the classification of simple Lie algebras into four great classes and five exceptional algebras is well known. This leads to a determination of the simple Lie algebras over arbitrary F of characteristic 0 which by now is almost complete, and in this sense we can say that all semi simple Lie algebras over F are known.

II. ARBITRARY NON ASSOCIATIVE ALGEBRAS

Turning now to arbitrary non associative algebras, I should mention first that the concepts of sub algebra, ideal, homomorphism, isomorphism, simple algebra, difference algebra, and direct sum do not involve associativity in any way. If a non associative algebra can be written as the direct sum S i 0 • • • ®Sr of simple two-sided ideals Si, it is easy to see that the simple summands in such a decomposition are uniquely determined. If we hope to use the associative and Lie theories as models at all, then we are forced to define a finite-dimensional non associative algebra to be semi simple in case it is a direct sum of simple two-sided ideals. (For some purposes it may be desirable to restrict the notion of simplicity, but that is a refinement which I can take up later.) The characteristic property of the radical then is not that it is the maximal nil-, solvable, or nilpotent ideal, but that it is the minimal ideal N such that A/N is zero or a direct sum of simple two-sided ideals [4]. Thus any non associative algebra A has a radical N, and it is easy to see that the difference algebra A/N is semi simple in the original sense that its radical is zero. Also A/N is uniquely expressible Sets with two binary operations ++ and ··, satisfying all the axioms of associative rings and algebras except possibly the associativity of multiplication. The first examples of non-associative rings and algebras that are not associative appeared in the mid-19th century (Cayley numbers and, in general, hyper complex numbers, cf. Hyper complex number). Given an associative ring (algebra), if one replaces the ordinary multiplication by the operation $[a,b]=ab-ba[a,b]=ab-ba$, the result is a non-associative ring (algebra) that is a Lie ring (algebra). Yet another important class of non-associative rings (algebras) is that of Jordan rings (algebras); these are obtained by defining the operation $a\cdot b=(ab+ba)/2a\cdot b=(ab+ba)/2$ in an associative algebra over a field of characteristic $\neq 2\neq 2$ (or over a commutative ring of operators with a 1 and a 1/21/2). The theory of non-associative rings and algebras has evolved into an independent branch of algebra, exhibiting many points of contact with other fields of mathematics and also with physics, mechanics, biology, and other sciences. The central part of the theory is the theory of what are known as nearly-associative rings and algebras: Lie, alternative, Jordan, Mal'tsev rings and algebras, and some of their generalizations (see Lie algebra; Alternative rings and algebras; Jordan algebra; Mal'tsev algebra).

One of the most important problems that must be solved when studying any class of non-associative algebras is the description of simple algebras, both finite dimensional and infinite dimensional. In this context, the word description is to be understood modulo some "classical" class contained in the class being described (e.g. the description of simple algebras in the class of alternative rings is given modulo associative rings; for Mal'tsev algebras — modulo Lie algebras; for Jordan algebras — modulo special Jordan algebras; etc.). From this point of view, the various classes of non-associative algebras can be divided into those in which there are "many" simple algebras and those in which there are "few". Typical classes in which there are many simple algebras are the associative algebras, the Lie algebras and the special Jordan algebras. Namely, in these classes the following imbedding theorem is valid: Any associative (Lie, special Jordan) algebra over a field can be imbedded in a simple algebra of the same type. In some classes of algebras there are much simple algebra that are far from associative in the class of all algebras and in the class of all commutative (anti-commutative) algebras. For these classes, too, there holds an imbedding theorem analogous to that cited above.

The problem of describing the finite-dimensional simple associative (Lie, alternative or Jordan) algebras is the object of the classical part of the theory of these algebras. Subsequently, the main results about the structure of simple finite-dimensional associative (alternative, Jordan) algebras were carried over to Artinian rings of the same type — rings with the minimum condition for one-sided ideals; in Jordan rings, one-sided ideals are replaced by quadratic ideals (see Jordan algebra).

Classes of algebras with "few" simple algebras are interesting. Typical examples are the classes of alternative, Mal'tsev or Jordan algebras. In the class of alternative algebras, modulo associative algebras the only simple algebras are the (eight-dimensional) Cayley–Dickson algebras over an associative-commutative centre. In the class of Mal'tsev algebras, modulo Lie algebras the only simple algebras are the (seven-dimensional) algebras (relative to the commutator operation [a,b] [a,b]) associated with the Cayley–Dickson algebras. In the class of Jordan algebras, modulo the special Jordan algebras the simple algebras are the (twelve-dimensional) Albert algebras over their associative centers (algebras of the series EE) (see Jordan algebra). In larger classes, such as those of right-alternative or binary Lie algebras, the description of simple algebras is as yet incomplete (1989). It is known that there exists no finite-dimensional simple binary Lie algebra over a field of characteristic 0 other than a Mal'tsev algebra, but it is not known whether this result is valid in the infinite-dimensional case. For right-alternative algebras it is known that, although all finite-dimensional simple algebras of this class are alternative, there exist infinite-dimensional simple right-alternative algebras that are not alternative. All simple algebras are associative for the so-called (γ,δ)(γ,δ)-algebras (provided (γ,δ)≠(1,1)(γ,δ)≠(1,1)); these algebras arise in a natural manner from the stipulation that the square of an ideal be an ideal. A description is known for all Jordan algebras with two generators: Any Jordan algebra with two generators is a special Jordan algebra (Shirshov's theorem). All Jordan division algebras have been described (modulo associative division algebras).

In the classes of alternative, Mal'tsev or Jordan algebras there is a description of all primary rings (i.e. algebras the groupoid of two-sided ideals of which does not contain a zero divisor), as follows. A primary alternative ring (with 1/31/3 in the commutative ring of operators) is either associative or a Cayley–Dickson ring. A primary non-degenerate Jordan algebras is either special or is an Albert ring (a Jordan ring is called an Albert ring if its associative centre ZZ consists of regular elements and if the algebra Z−1AZ−1A is a twenty-seven-dimensional Albert algebra over its centre Z−1ZZ−1Z).

In a certain sense, the opposite of a simple algebra or a primary algebra is nil algebra. For power-associative algebras (cf. Algebra with associative powers) that are not anti-commutative (such as associative, alternative, Jordan, etc., algebras), nil algebras are defined as algebras in which some power of each element equals zero; in

the case of anti-commutative algebras (i.e. algebras with the identity x2=0x2=0, such as Lie, Mal'tsev and binary Lie algebras), nil algebras are the same as Engel algebras, i.e. algebras satisfying a condition
$\forall x,y\exists((xy)\cdots y)n=0 .\forall x,y\exists((xy)\cdots y)\overbrace{}n=0$.
In alternative (including associative) algebras, any nil algebra of bounded index (i.e. with an identity xn=0xn=0) is locally nilpotent, and if it has no mm-torsion (i.e. mx=0⇒x=0mx=0⇒x=0) for m≤nm≤n, it is solvable (in the associative case — nilpotent). Shirshov's problem concerning the local nilpotency of Jordan nil algebras of bounded index has been solved affirmatively. It is not known (1989) whether there exists a simple associative nil ring.

In the case of Lie algebras, the problem of the local nilpotency of Engel Lie algebras is solved by Kostrikin's theorem: Any Lie algebra with an identity
$[\ldots[x,y],\ldots,y]n=0 .[\ldots[x,y],\ldots,y]\overbrace{}n=0$.
over a field of characteristic p>np>n is locally nilpotent. This theorem implies a positive solution to the restricted Burnside problem for groups of exponent pp. Recently, E.I. Zel'manov (1989) has proved the local nilpotency of Engel Lie algebras over a field of arbitrary characteristic. From this he has inferred a positive solution of the restricted Burnside problem for groups of arbitrary exponent nn (using the classification of the finite simple groups). In general, all problems connected with the local nilpotency of nil algebras are known as Burnside-type problems. Among these is also Kurosh problem concerning the local finiteness of algebraic algebras (cf. Algebraic algebra). An alternative (in particular, associative) algebraic algebra AA of bounded degree (i.e. the degrees of the polynomials satisfied by elements of AA are uniformly bounded) is locally finite. In the general case, however, Burnside-type problems (such as the local nilpotency of associative nil rings, etc.) have negative solutions.

Another topic of study includes free algebras and free products of algebras in various varieties. In the variety of all non-associative algebras, any sub algebra of a free algebra is free, and any sub algebra of a free product of algebras is the free product of its intersections with the factors and some free algebra (Kurosh theorem). Theorems of this type are also valid in varieties of commutative (anti-commutative) algebras. These questions are most interesting for Lie algebras. Any sub algebra of a free Lie algebra is itself a free Lie algebra (the Shirshov–Witt theorem). However, the analogue of Kurosh theorem is no longer valid for sub algebras of a free product of Lie algebras; nevertheless, such sub algebras may be described in terms of the generators of an ideal modulo which the free product of the intersections and the free sub algebra must be factorized. Research has been done on free alternative algebras — their Zhevlakov radicals (quasi-regular radicals, cf. Quasi-regular radical), their centers (associative and commutative), the quotient algebras modulo the Zhevlakov radical, etc. In contrast to free associative algebras, free alternative algebras with n≥4n≥4generators contain zero divisors and, moreover, trivial ideals (non-zero ideals with zero square). There are also known instances of trivial ideals in free Mal'tsev algebras with n≥5n≥5 generators; while concerning free Jordan algebras with n≥3n≥3 generators all that is known is that they contain zero divisors, nil elements and central elements. The theory of free algebras is closely bound up with questions of identities in various classes of algebras. In this connection one also has the problem of the basis rank of a variety (the basis rank is the smallest natural number nn such that the variety in question is generated by a free algebra with nn generators; if no such nn exists, the basis rank is defined as infinity). The basis rank of the varieties of associative and Lie algebras are 2; that of alternative and Mal'tsev algebras is infinite.

The general theory of varieties and classes of non-associative algebras deals with classes of algebras on the borderline of the classical ones and with their various relationships. One characteristic result is the following. It turns out that the varieties of admissible, generalized admissible and generalized standard algebras defined at different times and by different authors actually belong to the eight-element sublattice of the lattice of all varieties of non-associative algebras, which is also made up of the varieties of Jordan, commutative, associative, associative-commutative, and alternative algebras. The variety generated by a finite associative (alternative, Lie, Mal'tsev, or Jordan) ring is finitely based, while there exists a finite non-associative ring (an algebra over a finite field) that generates an infinitely based variety. There exists a Lie algebra over an infinite field with this property. At the same time, it is still (1989) not known whether there exists a non-finitely based variety of Lie algebras over a field of characteristic zero. A.R. Kemer [18] has proved that every variety of associative algebras over a field of characteristic 0 is finitely based (a positive solution to Specht's problem).

III. CONCLUSION:

The algorithmic problems in the theory of non-associative rings and algebras have been formulated under the influence of mathematical logic. It is known that the word problem in the variety of all non-associative algebras is solvable (Zhukov's theorem). An analogous result is valid for commutative (anti-commutative) algebras. It is known that the Lie algebras with one relation have a solvable word problem. At the same time, there exist finitely-presented Lie algebras with an unsolvable word problem. The word problem has also been investigated in the variety of solvable Lie algebras of a given solvability degree nn; it is solvable for n=2n=2, unsolvable for n≥3n≥3. It has been proved that any recursively-defined Lie algebra (associative algebra) over a prime field can be imbedded in a finitely-presented Lie algebra (associative algebra).

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