TIME DEPENDENT SOLUTION AND BUSY PERIOD ANALYSIS OF A BALKED FEEDBACK QUEUEING MODEL WITH INTERMITTENTLY AVAILABLE SERVER

P.C. GARG
Department of Statistics
Punjabi University, Patiala
(147002), Punjab, INDIA
pcgarg2k2@yahoo.co.in

NEELAM SINGLA
Department of Statistics
Punjabi University, Patiala
(147002), Punjab, INDIA
neelgagan2k3@yahoo.co.in

TANVI JINDIA
Department of Statistics
Punjabi University, Patiala
(147002), Punjab, INDIA
tanvijindia@gmail.com

ABSTRACT
An intermittently available server feedback queueing model with balking has been taken into consideration for its mathematical modeling under transient state. The system is analyzed under the assumption that arriving customers are processed by a single server under FIFO queue discipline and arrival process follows Poisson distribution. In addition to this, service times and availability times of the server are exponentially distributed. The time dependent solution and the busy period distribution of the model have been explored. Some special cases have also been discussed.
Keywords: *Feedback, balking, intermittently available server, generating function, exponential distribution, busy period distribution.*

INTRODUCTION
A queue involves arriving customers that wait to be served at the service facility for the service they seek. Queueing theory is concerned with the statistical description of the behavior of queues. In most of queueing problems, it's assumed that the server is spontaneously available. This type of assumption seems feasible only in the case when server is automated. In a manually operated service channel due to some factors the server may not be instantaneously available. This type of servers is known as intermittently available servers. In intermittently available systems servers are subject to random periods of unavailability. Here the server has a choice to start a fresh service immediately or to make an interruption. However, it is assumed that he completes in hand service before interruption. This measure provides a better perception about the behavior of the queueing system. White and Christie (1958) seem to be the first, who studied a queueing model in which the service is provided with interruptions. Garg & Kumari (2007) also studied a feedback queueing system with intermittently available server and the Laplace transformation of probability generating function of transient-state queue length probabilities is obtained. Indra and Vijay (2008) obtained transient solution of a two-state bulk arrival markovian queueing model with intermittently available server.

The concepts loss and feedback are introduced for customers only. After the completion of service if the customer is dissatisfied due to incomplete or inappropriate quality of service, then he can rejoin the queue for receiving another regular service. This is called feedback in queueing theory. Formulation of queues with feedback mechanism was first introduced by Takacs (1963). If the server is busy at the time of the arrival of customer, then due to impatient behavior of the customer, customer may or may not join the queue. This is known balking. In queueing theory it is also known as loss. An amazing example of the occurrence of balking and reneging is given by Ancker and Gafarian (1962). Haight (1957) first considered an M/M/1 queue with balking. Queueing systems with balking, reneging, or both have been studied by many researchers.

Gupta, Joshi and Tiwari (2016) analysis an M/D/1 queueing system with balking and the steady-state solution of the models is derived.

In queueing theory time dependent solutions are studied for analyzing system behavior over a time horizon because the steady state analysis is inappropriate in situations where the time horizon is finite. The steady state analysis does not reveal the complete picture of the system behavior because it ignores the short term and start-up effects. In many applications, steady state analysis of the system performance does not make any sense because the system may never attain the state of equilibrium. Therefore, transient solution is useful for obtaining optimal solutions leading to the control of the system.

In the present paper, a markovian queueing system with intermittently available server and balking has been discussed, where customers arrive according to Poisson process and served one by one on FIFO basis. All incoming customers are given first essential service whereas some of them may demand service for the second time. The transient state and the busy period distribution of the problem are obtained.

The queueing system investigated in this model is described by the following assumptions:

1. Arrivals Poisson with parameter λ.
2. When there are customers in the queue, availability time of intermittent server is exponentially distributed with parameter v. The server is available to an empty queue and its availability is also exponentially distributed with parameter v_0.
3. Service time is exponentially distributed with parameter μ.
4. Customers are taken for service in their order of arrival.
5. For the customers getting service first time the probability of rejoining the system is p and that of leaving the system is q so that p+q =1. However the customers will have to leave the system definitely after getting service for the second time.
6. The probability that the customers joins the service channel for the first time is assumed to be c_1 and that for the second time is c_2, so that $c_1 + c_2=1$.
7. The arriving customer joins the queue definitely if the number of customers in the system is less than 'k'. It may join with probability β or balks the queue with probability $(1-\beta)$ if the number of customers in the system is $\geq k$.
8. The waiting space is infinite.
9. The stochastic process involved, viz
    a) arrival of customers
    b) departure of customers
    c) Availability of server are statistically independent.

DEFINITIONS

$P_{n,B}^{(k)}(t)$ = Probability that there are n customers in the system at time t and the next customer is to depart for the first time or second time according as k=0 or 1 and server is busy in relation to the queue, i.e. either a customer is being served or else one is to be taken just then. $n \geq 1$

$Q_{0,B}^{(0)}(t)$ = Probability that there are zero customers in the system at time t and the next customer to depart for the first time and the server is busy with empty queue.

$P_{n,F}^{(k)}(t)$ = Probability that there are n customers in the system at time t and the next customer is to depart for the first time or second time according as k= 0 or 1 and the server is free in relation to the queue, i.e. neither a customer is being served nor is any to be taken at that instant. $n \geq 0$

$P_n(t) =$ Probability that there are n customers in the system at time t. $n \geq 0$

$$P_{n,B}(t) = P_{n,B}^{(0)}(t) + P_{n,B}^{(1)}(t) \qquad n \geq 1 \quad ; \qquad Q_{0,B}(t) = Q_{0,B}^{(0)}(t)$$

$$P_{n,F}(t) = P_{n,F}^{(0)}(t) + P_{n,F}^{(1)}(t) \qquad n \geq 0 \quad ;$$

$$P_n(t) = P_{n,B}(t) + P_{n,F}(t) + Q_{n,B}(t) \qquad n \geq 0 \qquad (1)$$

Initially, $P_{0,F}^{(0)}(0) = 1$ and $P_{0,B}^{(0)}(t) = P_{0,B}^{(1)}(t) = P_{0,F}^{(1)}(t) = Q_{0,B}^{(1)}(t) = 0$, $t \geq 0$

THE DIFFERENCE DIFFERENTIAL EQUATIONS DESCRIBING THE SYSTEM ARE

$$\frac{d}{dt} Q_{0,B}^{(0)}(t) = -\lambda Q_{0,B}^{(0)}(t) + \nu_0 P_{0,F}^{(0)}(t) \qquad (2)$$

$$\frac{d}{dt} P_{n,B}^{(0)}(t) = -(\lambda + \mu) P_{n,B}^{(0)}(t) + \lambda P_{n-1,B}^{(0)}(t) + \nu P_{n,F}^{(0)}(t) + \lambda Q_{0,B}^{(0)}(t)\delta_{n,1} \quad ,1 \leq n \leq k-1 \qquad (3)$$

$$\frac{d}{dt} P_{n,B}^{(0)}(t) = -(\lambda\beta + \mu) P_{n,B}^{(0)}(t) + [\lambda\beta + \lambda\delta_{n-k+1,1}(1-\beta)] P_{n-1,B}^{(0)}(t) + \nu P_{n,F}^{(0)}(t) \quad ,n \geq k \qquad (4)$$

$$\frac{d}{dt} P_{n,B}^{(1)}(t) = -(\lambda + \mu) P_{n,B}^{(1)}(t) + \lambda P_{n-1,B}^{(1)}(t) + \nu P_{n,F}^{(1)}(t) \quad ,1 \leq n \leq k-1 \qquad (5)$$

$$\frac{d}{dt} P_{n,B}^{(1)}(t) = -(\lambda\beta + \mu) P_{n,B}^{(1)}(t) + [\lambda\beta + \lambda\delta_{n-k+1,1}(1-\beta)] P_{n-1,B}^{(1)}(t) + \nu P_{n,F}^{(1)}(t) \quad ,n \geq k \qquad (6)$$

$$\frac{d}{dt} P_{0,F}^{(0)}(t) = -(\lambda + \nu_0) P_{0,F}^{(0)}(t) + \mu\left\{q P_{1,B}^{(0)}(t) + P_{1,B}^{(1)}(t)\right\} \qquad (7)$$

$$\frac{d}{dt} P_{n,F}^{(0)}(t) = -(\lambda + \nu) P_{n,F}^{(0)}(t) + \lambda P_{n-1,F}^{(0)}(t) + \mu c_1\left\{q P_{n+1,B}^{(0)}(t) + P_{n+1,B}^{(1)}(t)\right\} + \mu c_1 p P_{n,B}^{(0)}(t)(1 - \delta_{n,1}) \quad ,1 \leq n \leq k-1 \qquad (8)$$

$$\frac{d}{dt} P_{n,F}^{(0)}(t) = -(\lambda\beta + \nu) P_{n,F}^{(0)}(t) + [\lambda\beta + \lambda\delta_{n-k+1,1}(1-\beta)] P_{n-1,F}^{(0)}(t) + \mu c_1\left\{q P_{n+1,B}^{(0)}(t) + P_{n+1,B}^{(1)}(t)\right\} + \mu c_1 p P_{n,B}^{(0)}(t) \quad , \quad n \geq k \qquad (9)$$

$$\frac{d}{dt} P_{n,F}^{(1)}(t) = -(\lambda + \nu) P_{n,F}^{(1)}(t) + \lambda P_{n-1,F}^{(1)}(t) + \mu c_2\left\{q P_{n+1,B}^{(0)}(t) + P_{n+1,B}^{(1)}(t)\right\} + \mu(c_1\delta_{n,1} + c_2) p P_{n,B}^{(0)}(t) \quad ,1 \leq n \leq k-1 \qquad (10)$$

$$\frac{d}{dt} P_{n,F}^{(1)}(t) = -(\lambda\beta + \nu) P_{n,F}^{(1)}(t) + [\lambda\beta + \lambda\delta_{n-k+1,1}(1-\beta)] P_{n-1,F}^{(1)}(t) + \mu c_2\left\{q P_{n+1,B}^{(0)}(t) + P_{n+1,B}^{(1)}(t)\right\} + \mu c_2 p P_{n,B}^{(0)}(t) \quad , \quad n \geq k \qquad (11)$$

Where $\delta_{n,1} = \begin{cases} 1 & , \quad \text{for } n = 1 \\ 0 & , \quad \text{otherwise} \end{cases}$

Taking Laplace Transformation $\overline{P}_n(s) = \int_0^\infty e^{-st} P_n(t)\, dt$; ***Re*** $s > 0$ **of (2) - (11) and dividing by** μ

$$\left(\rho+\frac{s}{\mu}\right)Q_{0,B}^{(0)}=\left(\frac{\nu_0}{\mu}\right)P_{0,F}^{(0)}+\frac{1}{\mu} \tag{12}$$

$$\left(\rho+1+\frac{s}{\mu}\right)\bar{P}_{n,B}^{(0)}(s)=\rho\bar{P}_{n-1,B}^{(0)}(s)+\left(\frac{\nu}{\mu}\right)\bar{P}_{n,F}^{(0)}(s)+\rho\bar{Q}_{0,B}^{(0)}(s)\delta_{n,1} \qquad ,1\le n\le k-1 \tag{13}$$

$$\left(\rho\beta+1+\frac{s}{\mu}\right)\bar{P}_{n,B}^{(0)}(s)=\left[\rho\beta+\rho\delta_{n-k+1,1}(1-\beta)\right]\bar{P}_{n-1,B}^{(0)}(s)+\left(\frac{\nu}{\mu}\right)\bar{P}_{n,F}^{(0)}(s) \qquad ,n\ge k \tag{14}$$

$$\left(\rho+1+\frac{s}{\mu}\right)\bar{P}_{n,B}^{(1)}(s)=\rho\bar{P}_{n-1,B}^{(0)}(s)+\left(\frac{\nu}{\mu}\right)\bar{P}_{n,F}^{(1)}(s) \qquad ,1\le n\le k-1 \tag{15}$$

$$\left(\rho\beta+1+\frac{s}{\mu}\right)\bar{P}_{n,B}^{(1)}(s)=\left[\rho\beta+\rho\delta_{n-k+1,1}(1-\beta)\right]\bar{P}_{n-1,B}^{(1)}(s)+\left(\frac{\nu}{\mu}\right)\bar{P}_{n,F}^{(1)}(s) \qquad ,n\ge k \tag{16}$$

$$\left(\rho+\frac{\nu_0}{\mu}+\frac{s}{\mu}\right)\bar{P}_{0,F}^{(0)}(s)=\left\{q\bar{P}_{1,B}^{(0)}(s)+\bar{P}_{1,B}^{(1)}(s)\right\} \tag{17}$$

$$\left(\rho+\frac{\nu}{\mu}+\frac{s}{\mu}\right)\bar{P}_{n,F}^{(0)}(s)=\rho\bar{P}_{n-1,F}^{(0)}(s)+c_1\left\{q\bar{P}_{n+1,B}^{(0)}(s)+\bar{P}_{n+1,B}^{(1)}(s)\right\}+c_1p\left(1-\delta_{n,1}\right)\bar{P}_{n,B}^{(0)}(s) \qquad ,1\le n\le k-1 \tag{18}$$

$$\left(\rho\beta+\frac{\nu}{\mu}+\frac{s}{\mu}\right)\bar{P}_{n,F}^{(0)}(s)=\left[\rho\beta+\rho\delta_{n-k+1,1}(1-\beta)\right]\bar{P}_{n-1,F}^{(0)}(s)+c_1\left\{q\bar{P}_{n+1,B}^{(0)}(s)+\bar{P}_{n+1,B}^{(1)}(s)\right\}+c_1p\bar{P}_{n,B}^{(0)}(s) \qquad ,n\ge k \tag{19}$$

$$\left(\rho+\frac{\nu}{\mu}+\frac{s}{\mu}\right)\bar{P}_{n,F}^{(1)}(s)=\rho\bar{P}_{n-1,F}^{(1)}(s)+c_2\left\{q\bar{P}_{n+1,B}^{(0)}(s)+\bar{P}_{n+1,B}^{(1)}(s)\right\}+\left(c_1\delta_{n,1}+c_2\right)p\bar{P}_{n,B}^{(0)}(s) \qquad ,1\le n\le k-1 \tag{20}$$

$$\left(\rho+\frac{\nu}{\mu}+\frac{s}{\mu}\right)\bar{P}_{n,F}^{(1)}(s)=\left[\rho\beta+\rho\delta_{n-k+1,1}(1-\beta)\right]\bar{P}_{n-1,F}^{(1)}(s)+c_2\left\{q\bar{P}_{n+1,B}^{(0)}(s)+\bar{P}_{n+1,B}^{(1)}(s)\right\}+c_2p\bar{P}_{n,B}^{(0)}(s) \qquad ,n\ge k \tag{21}$$

DEFINITIONS

$$P_B{}^{(K)}(z,t)=\sum_{n=1}^{\infty}P_{n,B}^{(K)}(t)z^n+Q_{0,B}^{(0)}(t) \qquad P_F{}^{(K)}(z,t)=\sum_{n=0}^{\infty}P_{n,F}^{(K)}(t)\,z^n$$

$$P_B(z,t)=P_B{}^{(0)}(z,t)+P_B{}^{(1)}(z,t) \qquad P_F(z,t)=P_F{}^{(0)}(z,t)+P_F{}^{(1)}(z,t)$$

$$\bar{P}_B{}^{(K)}(z,s)=\int_0^{\infty}e^{-st}\,P_B{}^{(K)}(z,t)\,dt \qquad \bar{P}_F{}^{(K)}(z,s)=\int_0^{\infty}e^{-st}\,P_F{}^{(K)}(z,t)\,dt$$

$$P(z,t)=P_B(z,t)+P_F(z,t) \qquad \bar{P}(z,s)=\int_0^{\infty}e^{-st}\,P(z,t)\,dt$$

all for $k=0$ or 1 with $|z|\le 1$

Laplace Transformation of Probability Generating Function Of Transient-State Queue Length Probabilities

$$\bar{P}(z,s) = \frac{\begin{gathered}\frac{z}{\mu}\left(B(z)+\frac{\nu}{\mu}\right)\left\{A(z)-\frac{\nu}{\mu}c_2p(1-z)\right\} \\ -(1-z)\left(qA(z)+\frac{\nu}{\mu}c_2pz\right)\left\{\frac{(\nu-\nu_0}{\mu}\bar{P}_{0,F}^{(0)}(s)+C(z)\bar{Q}_{0,B}^{(0)}(s)\right\} \\ +z(1-z)\frac{\nu}{\mu}\left(B(z)+\frac{\nu}{\mu}\right)\left\{(c_1pz-c_2q)\bar{P}_{1,B}^{(0)}(s)+c_2\bar{P}_{1,B}^{(1)}(s)\right\} \\ +\left(1-z)(1-\beta)\right)\rho z\frac{\nu}{\mu}\left[\left\{A(z)-c_2p\frac{\nu}{\mu}(1-z)\right\}\sum_{n=0}^{k-1}\bar{P}_{n,F}^{(0)}(s)z^n \right. \\ \left. +\left\{A(z)+c_1p\frac{\nu}{\mu}(1-z)\right\}\sum_{n=0}^{k-1}\bar{P}_{n,F}^{(1)}(s)z^n\right] \\ -\left(1-z)(1-\beta)\right)\rho\left[\left\{\{A(z)(q+pz)+zC(z)\}-C(z)c_2pz\frac{\nu}{\mu}(1-z)\right\}\sum_{n=1}^{k-1}\bar{P}_{n,B}^{(0)}(s)z^n\right. \\ \left. +\left\{A(z)\{zC(z)+1\}+zC(z)c_1p\frac{\nu}{\mu}(1-z)\right\}\sum_{n=1}^{k-1}\bar{P}_{n,B}^{(1)}(s)z^n\right]\end{gathered}}{\left(A(z)-\frac{\nu}{\mu}c_2\right)\left\{(A(z)-\frac{\nu}{\mu}(q+pz)c_1)\right\}-\left(\frac{\nu}{\mu}\right)^2(q+pz)c_1c_2}$$

$$\rho = \lambda/\mu < 1\,; \quad |z| \le 1 \tag{22}$$

Where $B(z) = \left\{-\rho\beta z + \left(\rho\beta + 1 + \frac{s}{\mu}\right)\right\}, \quad C(z) = \left\{-\rho\beta z + \left(\rho\beta + \frac{\nu}{\mu} + \frac{s}{\mu}\right)\right\}$

$$A(z) = z.B(z).C(z) = \left\{-\rho\beta z^2 + \left(\rho\beta + 1 + \frac{s}{\mu}\right)z\right\} * \left\{-\rho\beta z + \left(\rho\beta + \frac{\nu}{\mu} + \frac{s}{\mu}\right)\right\}$$

Let denominator $D(z) = K_1(z) * K_2(z) - \left(\frac{\nu}{\mu}\right)^2 c_1c_2(q+pz)$

Where $K_1(z) = \left\{-\rho\beta z^2 + \left(\rho\beta + 1 + \frac{s}{\mu}\right)z\right\}\left\{-\rho\beta z + \left(\rho\beta + \frac{\nu}{\mu} + \frac{s}{\mu}\right)\right\} - c_2\frac{\nu}{\mu}$

$$K_2(z) = \left\{-\rho\beta z^2 + \left(\rho\beta + 1 + \frac{s}{\mu}\right)z\right\}\left\{-\rho\beta z + \left(\rho\beta + \frac{\nu}{\mu} + \frac{s}{\mu}\right)\right\} - c_1\frac{\nu}{\mu}(q+pz)$$

Obviously $K_1(z)$ and $K_2(z)$ have two zeroes inside the unit circle.

Let $f(z) = K_1(z) * K_2(z)$ and $g(z) = \left(\frac{\nu}{\mu}\right)^2 c_1c_2(q+pz)$

$|f(z)| = |K_1(z)| * |K_2(z)|$

$= \left|\left[\left\{-\rho\beta z^2 + \left(\rho\beta + 1 + \frac{s}{\mu}\right)z\right\}\left\{-\rho\beta z + \left(\rho\beta + \frac{\nu}{\mu} + \frac{s}{\mu}\right)\right\} - c_2\frac{\nu}{\mu}\right]\right|$

$$* \left\| \left[\left\{ -\rho\beta z^2 + \left(\rho\beta + 1 + \frac{s}{\mu}\right) z \right\} \left\{ -\rho\beta z + \left(\rho\beta + \frac{\nu}{\mu} + \frac{s}{\mu}\right) \right\} - c_1 \frac{\nu}{\mu}(q + pz) \right] \right\|$$

$$\geq \left(\xi + \left(\frac{\nu}{\mu}\right) c_2\right)\left(\xi + \left(\frac{\nu}{\mu}\right) c_1\right) \qquad \text{for } \frac{s}{\mu} = \xi + i\eta, \ |z| \leq 1$$

$$> \left(\frac{\nu}{\mu}\right)^2 c_1 c_2 \geq |g(z)|$$

Hence $|f(z)| > |g(z)|$ on $|z| \leq 1$

Since all the conditions of Rouche's Theorem are satisfied, so *D* has two zeroes inside the unit circle. Let these zeroes be z_m(m =*0,1*). Numerator must vanish for these two zeroes since $\bar{P}(z,s)$ is an analytical function of z. These two equations along with equation(17), (12), {(13) & (15) for *n=1,2*} and equation {(18) & (20) for *n=1*} (in case *k=3*) will determine the ten unknowns $\bar{P}_{0,F}^{(0)}(s)$, $\bar{P}_{1,F}^{(0)}(s)$, $\bar{P}_{1,F}^{(1)}(s)$, $\bar{P}_{1,B}^{(0)}(s)$, $\bar{P}_{1,B}^{(1)}(s)$, $\bar{P}_{2,F}^{(0)}(s)$, $\bar{P}_{2,F}^{(1)}(s)$,$\bar{P}_{2,B}^{(0)}(s)$, $\bar{P}_{2,B}^{(1)}(s)$ and $\bar{Q}_0^{(0)}(s)$. Along with these equations and equations (17), (12), {(13) & (15) for *n=1,2,3*} and equation {(18) and (20) for *n=1,2*} (in case *k=4*) will determine the fourteen unknowns $\bar{P}_{0,F}^{(0)}(s)$, $\bar{P}_{1,F}^{(0)}(s)$, $\bar{P}_{1,F}^{(1)}(s)$, $\bar{P}_{1,B}^{(0)}(s)$,$\bar{P}_{1,B}^{(1)}(s)$, $\bar{P}_{2,F}^{(0)}(s)$, $\bar{P}_{2,F}^{(1)}(s)$,$\bar{P}_{2,B}^{(0)}(s)$, $\bar{P}_{2,B}^{(1)}(s)$, $\bar{P}_{3,B}^{(0)}(s)$ $\bar{P}_{3,B}^{(1)}(s)$,$\bar{P}_{3,F}^{(0)}(s)$, $\bar{P}_{3,F}^{(1)}(s)$ and $\bar{Q}_0^{(0)}(s)$. In general for (17), (12), {(13) & (15) for *n=1,2,…, k-2*} and equation {(18) and (20) for *n=1,2,3,…,k-4*} in general determine *(4k-2)* unknowns $\bar{P}_{0,F}^{(0)}(s)$, $\bar{P}_{1,F}^{(0)}(s)$, $\bar{P}_{1,F}^{(1)}(s)$,$\bar{P}_{1,B}^{(0)}(s)$,$\bar{P}_{1,B}^{(1)}(s)$,$\bar{P}_{2,F}^{(0)}(s)$, … ,$\bar{P}_{k-1,F}^{(0)}(s)$, ,$\bar{P}_{k-1,F}^{(1)}(s)$(when number of customers= k).Hence the generating function $\bar{P}(z,s)$ is completely known. $\bar{P}_n(s)$ can be obtained by using the following formula.

SPECIAL CASE

1. **When there is no balking.**
   Put $\beta = 1$ in equation (22)

$$\bar{P}(z,s) = \frac{\begin{array}{l} \frac{z}{\mu}\left(B'(z) + \frac{\nu}{\mu}\right)\left\{A'(z) - \frac{\nu}{\mu}c_2 p(1-z)\right\} \\ - (1-z)\left(qA'(z) + \frac{\nu}{\mu}c_2 pz\right)\left\{\frac{(\nu - \nu_0}{\mu}\bar{P}_{0,F}^{(0)}(s) + C'(z)\bar{Q}_{0,B}^{(0)}(s)\right\} \\ + z(1-z)\frac{\nu}{\mu}\left(B'(z) + \frac{\nu}{\mu}\right)\left\{(c_1 pz - c_2 q)\bar{P}_{1,B}^{(0)}(s) + c_2\bar{P}_{1,B}^{(1)}(s)\right\} \end{array}}{\left(A'(z) - \frac{\nu}{\mu}c_2\right)\left\{(A'(z) - \frac{\nu}{\mu}(q+pz)c_1)\right\} - \left(\frac{\nu}{\mu}\right)^2 (q+pz)c_1 c_2}$$

Where $B'(z) = \left\{-\rho z + \left(\rho + 1 + \frac{s}{\mu}\right)\right\}$ $\qquad C'(z) = \left\{-\rho z + \left(\rho\beta + \frac{\nu}{\mu} + \frac{s}{\mu}\right)\right\}$

$$A'(z) = z.B'(z).C^{'}(z) = \left\{-\rho z^2 + \left(\rho + 1 + \frac{s}{\mu}\right)z\right\}\left\{-\rho z + \left(\rho + \frac{\nu}{\mu} + \frac{s}{\mu}\right)\right\}$$

2. **When the server is available to an empty queue with same mean availability time as is to a non-empty queue**
   Put $\nu_0 = \nu$ in equation (22), probability generating function for the queue length probabilities is

$$\bar{P}(z,s) = \frac{\begin{aligned}&\frac{z}{\mu}\left(B'(z) + \frac{\nu}{\mu}\right)\left\{A'(z) - \frac{\nu}{\mu}c_2 p(1-z)\right\} - (1-z)\left(qA'(z) + \frac{\nu}{\mu}c_2 pz\right)\left\{C'(z)\bar{Q}_{0,B}^{(0)}(s)\right\} \\ &+ z(1-z)\frac{\nu}{\mu}\left(B'(z) + \frac{\nu}{\mu}\right)\left\{(c_1 pz - c_2 q)\bar{P}_{1,B}^{(0)}(s) + c_2\bar{P}_{1,B}^{(1)}(s)\right\} \\ &+ \big(1-z)(1-\beta)\big)\rho z\frac{\nu}{\mu}\left[\left\{A'(z) - c_2 p\frac{\nu}{\mu}(1-z)\right\}\sum_{n=0}^{k-1}\bar{P}_{n,F}^{(0)}(s)z^n \right. \\ &\left. + \left\{A'(z) + c_1 p\frac{\nu}{\mu}(1-z)\right\}\sum_{n=0}^{k-1}\bar{P}_{n,F}^{(1)}(s)z^n\right] \\ &-\big(1-z)(1-\beta)\big)\rho\left[\left\{\{A'(z)(q+pz) + zC'(z)\} - C'(z)c_2 pz\frac{\nu}{\mu}(1-z)\right\}\sum_{n=1}^{k-1}\bar{P}_{n,B}^{(0)}(s)z^n \right. \\ &\left. + \left\{A'(z)\{zC^{'(z)} + 1\} + zC'(z)c_1 p\frac{\nu}{\mu}(1-z)\right\}\sum_{n=1}^{k-1}\bar{P}_{n,B}^{(1)}(s)z^n\right]\end{aligned}}{\left(A'(z) - \frac{\nu}{\mu}c_2\right)\left\{(A'(z) - \frac{\nu}{\mu}(q+pz)c_1)\right\} - \left(\frac{\nu}{\mu}\right)^2(q+pz)c_1 c_2}$$

$$\rho = \lambda/\mu < 1\,; \quad |z| \le 1$$

BUSY PERIOD DISTRIBUTION

The density function for the busy period distribution is given by

$$\frac{d}{dt}\left\{P_{0,F}^{(0)} + Q_{0,B}^{(0)}(t)\right\}$$

Initially, $P_{1,F}^{(0)}(0) = 1$ *and* $P_{0,F}^{(0)}(0) = Q_{0,B}^{(0)}(0) = P_{0,B}^{(1)}(t) = P_{0,F}^{(1)}(t) = 0\ ,\quad t \ge 0$

The difference differential equations describing the system are

$$\frac{d}{dt}P_{0,F}^{(0)}(t) = \mu\left\{qP_{1,B}^{(0)}(t) + P_{1,B}^{(1)}(t)\right\} \tag{23}$$

$$\frac{d}{dt}Q_{0,B}^{(0)}(t) = 0 \tag{24}$$

$$\frac{d}{dt}P_{1,B}^{(0)}(t) = -(\lambda + ⍰)P_{1,B}^{(0)}(t) + \nu P_{1,F}^{(0)}(t) \tag{25}$$

$$\frac{d}{dt}P_{1,F}^{(0)}(t) = -(\lambda + \nu)P_{1,F}^{(0)}(t) + \mu c_1\left\{qP_{2,B}^{(0)}(t) + P_{2,B}^{(1)}(t)\right\} \tag{26}$$

$$\frac{d}{dt}P_{n,B}^{(0)}(t) = -(\lambda + \mu)P_{n,B}^{(0)}(t) + \lambda P_{n-1,B}^{(0)}(t) + \nu P_{n,F}^{(0)}(t) \qquad ,2 \le n \le k-1 \qquad (27)$$

$$\frac{d}{dt}P_{n,B}^{(0)}(t) = -(\lambda\beta + \mu)P_{n,B}^{(0)}(t) + \left[\lambda\beta + \lambda\delta_{n-k+1,1}(1-\beta)\right]P_{n-1,B}^{(0)}(t) + \nu P_{n,F}^{(0)}(t) \ ,n \ge k \qquad (28)$$

$$\frac{d}{dt}P_{n,B}^{(1)}(t) = -(\lambda + \mu)P_{n,B}^{(1)}(t) + \lambda P_{n-1,B}^{(1)}(t) + \nu P_{n,F}^{(1)}(t) \qquad ,1 \le n \le k-1 \qquad (29)$$

$$\frac{d}{dt}P_{n,B}^{(1)}(t) = -(\lambda\beta + \mu)P_{n,B}^{(1)}(t) + \left[\lambda\beta + \lambda\delta_{n-k+1,1}(1-\beta)\right]P_{n-1,B}^{(1)}(t) + \nu P_{n,F}^{(1)}(t) \ ,n \ge k \qquad (30)$$

$$\frac{d}{dt}P_{n,F}^{(0)}(t) = -(\lambda + \nu)P_{n,F}^{(0)}(t) + \lambda P_{n-1,F}^{(0)}(t) + \mu c_1\left\{qP_{n+1,B}^{(0)}(t) + P_{n+1,B}^{(1)}(t)\right\}$$
$$+\mu c_1 p P_{n,B}^{(0)}(t)\left(1-\delta_{n,1}\right) \qquad ,1 \le n \le k-1 \qquad (31)$$

$$\frac{d}{dt}P_{n,F}^{(0)}(t) = -(\lambda\beta + \nu)P_{n,F}^{(0)}(t) + \left[\lambda\beta + \lambda\delta_{n-k+1,1}(1-\beta)\right]P_{n-1,F}^{(0)}(t)$$
$$+\mu c_1\left\{qP_{n+1,B}^{(0)}(t) + P_{n+1,B}^{(1)}(t)\right\} + \mu c_1 p P_{n,B}^{(0)}(t) \qquad ,n \ge k \qquad (32)$$

$$\frac{d}{dt}P_{n,F}^{(1)}(t) = -(\lambda + \nu)P_{n,F}^{(1)}(t) + \lambda P_{n-1,F}^{(1)}(t) + \mu c_2\left\{qP_{n+1,B}^{(0)}(t) + P_{n+1,B}^{(1)}(t)\right\}$$
$$+ \mu\left(c_1\delta_{n,1} + c_2\right)pP_{n,B}^{(0)}(t) \qquad ,1 \le n \le k-1 \qquad (33)$$

$$\frac{d}{dt}P_{n,F}^{(1)}(t) = -(\lambda\beta + \nu)P_{n,F}^{(1)}(t) + \left[\lambda\beta + \lambda\delta_{n-k+1,1}(1-\beta)\right]P_{n-1,F}^{(1)}(t)$$
$$+\mu c_2\left\{qP_{n+1,B}^{(0)}(t) + P_{n+1,B}^{(1)}(t)\right\} + \mu c_2 p P_{n,B}^{(0)}(t) \qquad ,n \ge k \qquad (34)$$

Taking the Laplace Transformation $\overline{\mathrm{P}}_{\mathrm{n}}(\mathrm{s}) = \int_0^\infty \mathrm{e}^{-\mathrm{st}}\,\mathrm{P}_{\mathrm{n}}(\mathrm{t})\,\mathrm{dt}$ **; Re** $\mathrm{s} > 0$ **of (2) - (13) and dividing by** μ

$$\left(\rho + 1 + \frac{s}{\mu}\right)\bar{P}_{n,B}^{(0)}(s) = \rho\bar{P}_{n-1,B}^{(0)}(s) + \left(\frac{\nu}{\mu}\right)\bar{P}_{n,F}^{(0)}(s) \qquad ,1 \le n \le k-1 \qquad (35)$$

$$\left(\rho\beta + 1 + \frac{s}{\mu}\right)\bar{P}_{n,B}^{(0)}(s) = \left[\rho\beta + \rho\delta_{n-k+1,1}(1-\beta)\right]\bar{P}_{n-1,B}^{(0)}(s) + \left(\frac{\nu}{\mu}\right)\bar{P}_{n,F}^{(0)}(s) \qquad ,n \ge k \qquad (36)$$

$$\left(\rho + 1 + \frac{s}{\mu}\right)\bar{P}_{n,B}^{(1)}(s) = \rho\bar{P}_{n-1,B}^{(0)}(s) + \left(\frac{\nu}{\mu}\right)\bar{P}_{n,F}^{(1)}(s) \qquad ,1 \le n \le k-1 \qquad (37)$$

$$\left(\rho\beta + 1 + \frac{s}{\mu}\right)\bar{P}_{n,B}^{(1)}(s) = \left[\rho\beta + \rho\delta_{n-k+1,1}(1-\beta)\right]\bar{P}_{n-1,B}^{(1)}(s) + \left(\frac{\nu}{\mu}\right)\bar{P}_{n,F}^{(1)}(s) \qquad ,n \ge k \qquad (38)$$

$$\left(\rho + \frac{\nu}{\mu} + \frac{s}{\mu}\right)P_{1,F}^{(0)}(s) = \frac{1}{\mu} + c_1\left\{qP_{2,B}^{(0)}(s) + P_{2,B}^{(1)}(s)\right\} \qquad (39)$$

$$\left(\rho + \frac{\nu}{\mu} + \frac{s}{\mu}\right)\bar{P}_{n,F}^{(0)}(s) = \rho\bar{P}_{n-1,F}^{(0)}(s) + c_1\left\{q\bar{P}_{n+1,B}^{(0)}(s) + \bar{P}_{n+1,B}^{(1)}(s)\right\} + c_1 p\bar{P}_{n,B}^{(0)}(s) \qquad (40)$$

$$\left(\rho\beta + \frac{\nu}{\mu} + \frac{s}{\mu}\right)\bar{P}_{n,F}^{(0)}(s) = \left[\rho\beta + \rho\delta_{n-k+1,1}(1-\beta)\right]\bar{P}_{n-1,F}^{(0)}(s) + c_1\left\{q\bar{P}_{n+1,B}^{(0)}(s) + \bar{P}_{n+1,B}^{(1)}(s)\right\}$$

$$+c_1p\bar{P}^{(0)}_{n,B}(s) \qquad ,n \geq k \qquad (41)$$

$$\left(\rho+\frac{\nu}{\mu}+\frac{s}{\mu}\right)\bar{P}^{(1)}_{n,F}(s)=\rho\bar{P}^{(1)}_{n-1,F}(s)+c_2\left\{q\bar{P}^{(0)}_{n+1,B}(s)+\bar{P}^{(1)}_{n+1,B}(s)\right\}+\left(c_1\delta_{n,1}+c_2\right)p\bar{P}^{(0)}_{n,B}(s)$$

$$1 \leq n \leq k-1 \qquad (42)$$

$$\left(\rho+\frac{\nu}{\mu}+\frac{s}{\mu}\right)\bar{P}^{(1)}_{n,F}(s)=\left[\rho\beta+\rho\delta_{n-k+1,1}(1-\beta)\right]\bar{P}^{(1)}_{n-1,F}(s)+c_2\left\{q\bar{P}^{(0)}_{n+1,B}(s)+\bar{P}^{(1)}_{n+1,B}(s)\right\}$$

$$+c_2p\bar{P}^{(0)}_{n,B} \qquad n \geq k \qquad (43)$$

DEFINITIONS

$$G_B{}^{(K)}(z,t)=\sum_{n=1}^{\infty}P^{(K)}_{n,B}(t)z^n \qquad G_F{}^{(K)}(z,t)=\sum_{n=1}^{\infty}P^{(K)}_{n,F}(t)\,z^n$$

$$G_B(z,t)=G_B{}^{(0)}(z,t)+G_B{}^{(1)}(z,t) \qquad G_F(z,t)=G_F{}^{(0)}(z,t)+G_F{}^{(1)}(z,t)$$

$$G^{(K)}(z,t)=G_B{}^{(K)}(z,t)+G_F{}^{(K)}(z,t) \qquad G(z,t)=G^{(0)}(z,t)+G^{(1)}(z,t)$$

$$G(z,t)=G_B(z,t)+G_F(z,t) \qquad all\ for\ k=0\ or\ 1\ with\ |z|\leq 1$$

Laplace Transformation of Probability Generating Function for the Busy Period Distribution

$$\bar{G}(z,s)=\frac{\begin{gathered}\frac{z^2}{\mu}\left(B(z)+\frac{\nu}{\mu}\right)\left\{A(z)-\frac{\nu}{\mu}c_2p(1-z)\right\}\\ -z\left(B(z)+\frac{\nu}{\mu}\right)\left\{\left\{qA(z)-\frac{\nu}{\mu}c_1zp^2(1-z)\right)\bar{P}^{(0)}_{1,B}(s)\right\}+A(z)\bar{P}^{(1)}_{1,B}(s)\right\}\\ -(1-z)(1-\beta))\rho z^2\left(B(z)+\frac{\nu}{\mu}\right)\left[\left\{A(z)-c_2p\frac{\nu}{\mu}(1-z)\right\}\sum_{n=1}^{k-1}\bar{P}^{(0)}_{n,F}(s)z^n\right.\\ \left.+\left\{A(z)+c_1p\frac{\nu}{\mu}(1-z)\right\}\sum_{n=1}^{k-1}\bar{P}^{(1)}_{n,F}(s)z^n\right]\\ -(1-z)(1-\beta))\rho z^2C(z)\left[\left\{\{B(z)(q+pz)+A(z)\}-c_2p\frac{\nu}{\mu}(1-z)\right\}\sum_{n=1}^{k-1}\bar{P}^{(0)}_{n,B}(s)z^n\right.\\ \left.-\left\{A(z)+B(z)+c_1p\frac{\nu}{\mu}(1-z)\right\}\sum_{n=1}^{k-1}\bar{P}^{(1)}_{n,B}(s)z^n\right]\end{gathered}}{\left(A(z)-\frac{\nu}{\mu}c_2\right)\left\{(A(z)-\frac{\nu}{\mu}(q+pz)c_1)\right\}-\left(\frac{\nu}{\mu}\right)^2(q+pz)c_1c_2}$$

$$\lambda/\mu<1\,;\ \ |z|\leq 1 \qquad (44)$$

Where $B(z)=\left\{-\rho\beta z+\left(\rho\beta+1+\frac{s}{\mu}\right)\right\}$ $\qquad C(z)=\left\{-\rho\beta z+\left(\rho\beta+\frac{\nu}{\mu}+\frac{s}{\mu}\right)\right\}$

$$A(z) = z.B(z).C(z) = \left\{-\rho\beta z^2 + \left(\rho\beta + 1 + \frac{s}{\mu}\right)z\right\}\left\{-\rho\beta z + \left(\rho\beta + \frac{\nu}{\mu} + \frac{s}{\mu}\right)\right\}$$

Let $D(z) = K_1(z) * K_2(z) - \left(\frac{\nu}{\mu}\right)^2 c_1 c_2 (q + pz)$

Where $K_1(z) = \left\{-\rho\beta z^2 + \left(\rho\beta + 1 + \frac{s}{\mu}\right)z\right\}\left\{-\rho\beta z + \left(\rho\beta + \frac{\nu}{\mu} + \frac{s}{\mu}\right)\right\} - c_2\frac{\nu}{\mu}$

$$K_2(z) = \left\{-\rho\beta z^2 + \left(\rho\beta + 1 + \frac{s}{\mu}\right)z\right\}\left\{-\rho\beta z + \left(\rho\beta + \frac{\nu}{\mu} + \frac{s}{\mu}\right)\right\} - c_1\frac{\nu}{\mu}(q + pz)$$

Obviously $K_1(z)$ and $K_2(z)$ have two zeroes inside the unit circle.

Let $f(z) = K_1(z) * K_2(z)$ and $g(z) = \left(\frac{\nu}{\mu}\right)^2 c_1 c_2 (q + pz)$

$$|f(z)| = |K_1(z)| * |K_2(z)|$$

$$= \left|\left[\left\{-\rho\beta z^2 + \left(\rho\beta + 1 + \frac{s}{\mu}\right)z\right\}\left\{-\rho\beta z + \left(\rho\beta + \frac{\nu}{\mu} + \frac{s}{\mu}\right)\right\} - c_2\frac{\nu}{\mu}\right]\right|$$

$$* \left|\left[\left\{-\rho\beta z^2 + \left(\rho\beta + 1 + \frac{s}{\mu}\right)z\right\}\left\{-\rho\beta z + \left(\rho\beta + \frac{\nu}{\mu} + \frac{s}{\mu}\right)\right\} - c_1\frac{\nu}{\mu}(q + pz)\right]\right|$$

$$\geq \left(\xi + \left(\frac{\nu}{\mu}\right)c_2\right)\left(\xi + \left(\frac{\nu}{\mu}\right)c_1\right) \qquad for \quad \frac{s}{\mu} = \xi + i\eta\,, \ |z| = 1$$

$$> \left(\frac{\nu}{\mu}\right)^2 c_1 c_2 \geq |g(z)|$$

Hence $|f(z)| > |g(z)|$ on $|z| = 1$

These two equations along with equation (39), {(35) & (37) for n=1,2 and (42) for n=1 } (in case k=3) will determine the eight unknowns $\bar{P}_{1,F}^{(0)}(s)$, $\bar{P}_{1,F}^{(1)}(s)$,$\bar{P}_{1,B}^{(0)}(s)$,$\bar{P}_{1,B}^{(1)}(s)$,$\bar{P}_{2,F}^{(0)}(s)$, $\bar{P}_{2,F}^{(1)}(s)$,$\bar{P}_{2,B}^{(0)}(s)$, $\bar{P}_{2,B}^{(1)}(s)$. Along with these equations and equations (39), {(35) & (37) for n=1,2,3 and (40) for *n=2* & (42) for *n=1,2*} (in case *k=4*) will determine the twelve unknowns $\bar{P}_{1,F}^{(0)}(s)$, $\bar{P}_{1,F}^{(1)}(s)$, $\bar{P}_{1,B}^{(0)}(s)$, $\bar{P}_{1,B}^{(1)}(s)$, $\bar{P}_{2,F}^{(0)}(s)$, $\bar{P}_{2,F}^{(1)}(s)$, $\bar{P}_{2,B}^{(0)}(s)$, $\bar{P}_{2,B}^{(1)}(s)$,$\bar{P}_{3,B}^{(0)}(s)$, $\bar{P}_{3,B}^{(1)}(s)$, $\bar{P}_{3,F}^{(0)}(s)$, $\bar{P}_{3,F}^{(1)}(s)$. Equation (39), {(35) & (37) for *n=1,2,3,…, k-2* and (40) for *n=2, 3,…,k-4* & (42) for *n=1,2, 3,…,k-4*} in general determine (4k-4) unknowns $\bar{P}_{1,F}^{(0)}(s)$, $\bar{P}_{1,F}^{(1)}(s)$, $\bar{P}_{1,B}^{(0)}(s)$, $\bar{P}_{1,B}^{(1)}(s)$, $\bar{P}_{2,F}^{(0)}(s)$, … , $\bar{P}_{k-1,F}^{(0)}(s)$, $\bar{P}_{k-1,F}^{(1)}(s)$, (when number of customers= k). Hence the generating function $\bar{G}(z,s)$ is completely known. The density function of busy period distribution $\frac{d}{dt}\left\{P_{0,F}^{(0)} + Q_{0,B}^{(0)}(t)\right\}$ can be obtained.

SPECIAL CASE

1. **When there is no balking i.e. put $\beta = 1$ in equation (44)**

$$\bar{G}(z,s)=\frac{\frac{z^2}{\mu}\left(B"(z)+\frac{\nu}{\mu}\right)\left\{A"(z)-\frac{\nu}{\mu}c_2p(1-z)\right\} - z\left(B"(z)+\frac{\nu}{\mu}\right)\left\{\{qA"(z)-\frac{\nu}{\mu}c_1zp^2(1-z))\bar{P}_{1,B}^{(0)}(s)\}+A"(z)\bar{P}_{1,B}^{(1)}(s)\right\}}{\left(A"(z)-\frac{\nu}{\mu}c_2\right)\left\{(A"(z)-\frac{\nu}{\mu}(q+pz)c_1)\right\}-\left(\frac{\nu}{\mu}\right)^2(q+pz)c_1c_2}$$

$$\rho=\lambda/\mu<1\,;\ |z|\leq 1 \qquad (45)$$

$$B"(z)=\left\{-\rho z+\left(\rho+1+\frac{s}{\mu}\right)\right\} \qquad C"(z)=\left\{-\rho z+\left(\rho\beta+\frac{\nu}{\mu}+\frac{s}{\mu}\right)\right\}$$

$$A"(z)=z.B"(z).C"(z)=\left\{-\rho z^2+\left(\rho+1+\frac{s}{\mu}\right)z\right\}\left\{-\rho z+\left(\rho+\frac{\nu}{\mu}+\frac{s}{\mu}\right)\right\}$$

ACKNOWLEDGMENT
The authors are also thankful to Dr. Upasana Sharma, Associate Professor and Head, Department of Statistics, Punjabi University, Patiala for providing the necessary facilities needed for this work.

REFERENCES

1. *Ancker Jr., C. and Gafarian, A. Queueing with impatient customers who leave at random. Journal of Industry Engineering, (1962), XIII, 84–90.*
2. *Finch, P.D.,Cyclic queues with feedback, Journal of Royal Statistical Society Series B-21,(1959), 153-157.*
3. *Garg and Singla, An Intermittently Available Server Queueing System with Feedback. Int. J. Agricult. Stat. Sci.,(2007), 3(2), 471-480*
4. *Gupta, V. K., Joshi TabiNandan and Tiwari, S.K, A Balked M/D/1 Feedback Queueing Models, IOSR Journal of Mathematics,(2016),Vol12,Issue 1 Vol. III, 83-87*
5. *Haight, F. A. , Queueing with balking, -I., Biometrika, (1957), Vol. 44, 360-369.*
6. *Indra and Kumar, Vijay ,Transient solution of a two-state Markovian queueing model with intermittently available server and arrivals in batches of variable size, International journal of Agriculture and Statistical Sciences, (2008),Vol. 4, No. 1, 97-106*
7. *Jindia, Garg &Singla,A Single Server Queueing Model with Feedback and Balking in an international journal, Journal of Rajasthan Academy of Physical Sciences, (2013), Vol. 12, No. 4, Dec. 2013, 373-384.*
8. *Krishna Kumar, B. Partha sarathy, P. R. and Sharafali, M.,Transient solution of an M/M/1 queue with balking, Queueing System, (1993), Vol. 13, 441-447.*
9. *Takacs, L. ,A Single Server Queue with Feedback, The Bell System Tech. Journal, .(1963), Vol. 42, 134-149.*
10. *White, H. and Christie, L.S. ,Queueing with Preemptive priorities or with breakdown, Operations Research,(1958),Vol. 6,79-96*