INDEPENDENT SET OF DIVISOR CAYLEY GRAPH

Dr. K. Sujatha
Head & Associate Professor, Department of Mathematics,
Tara Government College(A), Sanga Reddy, Sanga Reddy District -502001, Telangana, India
Affiliation: Osmania University, E-mail: sujatha.kudum@gmail.com

Abstract: *Let $n \geq 1$ be an integer and let S be the set of divisors of n other than n. Then the set $S^* = \{s, n - s / s \in S\}$ is a symmetric subset of the group $(Z_n, \oplus)$, the additive abelian group of integers modulo n. The Cayley graph of $(Zn, \oplus)$ associated with the above symmetric subset $S*$ is called the divisor Cayley graph and it is denoted by $G(Zn, D)$. That is, the graph $G(Zn, D)$ is the graph whose vertex set is $V = \{0,1,2,., n-1\}$ and the edge set E is the set of all ordered pairs of vertices x, y such that either $x - y \in S^*$, or , $y - x \in S^*$.*
In this paper we study the independent set, independence number and chromatic number of the divisor Cayley graph $G(Zn, D)$.

Key words: Divisor function, Cayley graph, independent set, independence number and chromatic number.

1. Introduction

Nathanson [10] was the pioneer in introducing the concepts of number theory and thus paved way for the study of a new class of graphs called Arithmetic Graphs. The theory of groups provides an interesting and powerful abstract approach to the study of symmetries of various graphs and one such tool is the notion of Cayley graph $G(X, S)$, associated with a group X and a symmetric subset S of X, whose vertex set is X and the edge set is the $\{(x, sx) / x \in X, s \in S\}$. Basically, the Cayley graphs are connected and vertex-transitive. If the group $(X,\cdot)$ is the additive group $(Z_n,\oplus)$ of integers 0,1,2, …, n-1 modulo n, and the symmetric set S is associated with some arithmetic function, then the Cayley graph may be treated as an arithmetic graph and such graphs are called arithmetic Cayley graphs. A new class of graphs namely, Cayley Graphs can be constructed by making use of a group X and a symmetric subset S of X (a subset S of X is called a symmetric subset if $s \in S \Rightarrow s^{-1} \in S$). It is the graph $G(X, S)$, whose vertex set is X and edge set $E = \{(x, y)/x, y \in X \text{ and either } xy^{-1} \in S, \text{ or}, yx^{-1} \in S\}$. It is well known that [Th. 1.4.5, p 16 of 8] $G(X, S)$ is an undirected graph without loops, which is $|S^*|$ − regular having $\frac{|X||S|}{2}$ edges.

Madhavi [9] studied the arithmetic Cayley graphs associated with Euler- totient function, Quadratic residue function modulo a prime and the divisor function. It is said that a precise notion of a dominating set, that is present in the current literature is given by Berge [4], Ore [11]. Later Harary and Livington [8], Cockayne and Hedetniemi [6] and many others have contributed significantly to this theory.

In this paper, the author studied the independent set, independence number and chromatic number of the divisor Cayley graph $G(Z_n, D)$. We refer the reader for graph theoretic notions Bondy and Murty [5] and Harary [7] and for number theoretic notions Apostol [2].

2. Divisor Cayley Graph

Definition 2.1: Let $n \geq 1$ be an integer and let S be the set of divisors of n other than n. Then the set $S^* = \{s, n - s / s \in S\}$ is a symmetric subset of the group $(Z_n, \oplus)$, the additive abelian group of integers modulo n. The Cayley graph of $(Z_n, \oplus)$ associated with the above symmetric subset S^* is called the **divisor Cayley graph** and it is denoted by $G(Z_n, D)$. That is, $G(Z_n, D)$ is the graph whose vertex set $V = \{0, 1, 2, \dots, n - 1\}$ and the edge set E is the set of all ordered pairs of vertices x, y such that either $x - y \in S^*$, or, $y - x \in S^*$.

Madhavi [6] introduced this graph and studied various properties of this graph. We list below some of them.

i. The graph $G(Z_n, D)$ is $|S^*|$ - regular, and the number of edges in $G(Z_n, D)$ is $\frac{n|S^*|}{2}$.

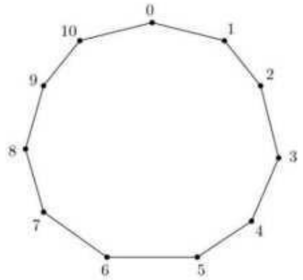
ii. The graph $G(Z_n, D)$ is Hamiltonian and hence connected.

iii. Degree of each vertex in $G(Z_n, D)$ is determined.

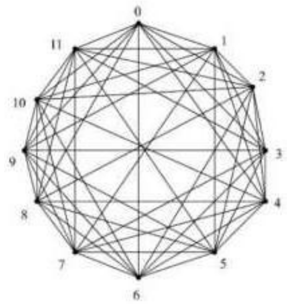
iv. The graph $G(Z_n, D)$ is not bipartite but Eulerian.

v. If n is a prime, then the graph $G(Z_n, D)$ reduces to the outer Hamilton cycle.

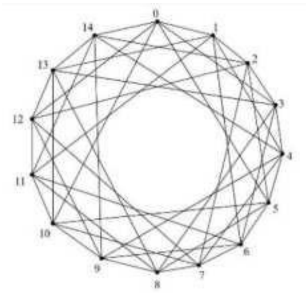
The graphs $G(Z_n, D)$ for $n = 11, 12$ and 15 are given below.



$G(Z_{11}, D)$



$G(Z_{12}, D)$



$G(Z_{15}, D)$

3. Main Results

Definition :3.1

A subset S of V is called an **independent set** of a graph G if no two vertices of S are adjacent in G.

Definition: 3.2

The number of vertices in the largest independent set of a graph G is called the **independence number** and it is denoted by α (G).

A single vertex of any graph G constitutes an independent set

Theorem: 3.3

Let $n = 5$, or, $n \geq 7$ and let d_0 be the smallest positive integer that does not divide n. For any integer $t \geq 1$, the set of vertices V_t in $G(Z_n, D)$, which is of the form $V_t = \{ t + rd_o / o \leq r \leq k - 1$, where k is the least positive integer such that $rd_o \notin s^*$ and $t + rd_0 < n \}$, is an independent set of $G(Z_n,D)$.

Proof :

Let n = 5, or, $n \geq 7$ and let d_0 be the smallest positive integer that does not divide n. For any integer $t \geq 1$, consider the set $V_t = \{ t + rd_0 / 0 \leq r \leq k - 1$, where k is the least positive integer such that $rd_0 \notin s^*$ and $t + rd_0 < n \}$.

Let $t + ld_0$, $t + md_0$ be any two distinct vertices in V_t. Then $l \neq m$. For definiteness let us take $m < l$. Then l, $m \leq k - 1$, so that $l - m < k - 1$. So by the definition of V_t, $(l - m) d_0 \notin S^*$. Now $(t + ld_0) - (t + md_0) = (l - m) d_0 \notin S^*$. So, $t + l d_0$ and $t + md_0$, $l \neq m$ are not adjacent and thus V_t is an independent set.

Next, let s be the smallest integer such that $t < s$ and $s \notin V_t$. Consider the set $V_s = \{ s + rd_0 / 0 \leq r \leq k - 1$, where k is the least positive integer such that $rd_0 \notin S^*$ and $s + rd_0 < n \}$. As in the case of V_t, one can show that V_s is also an independent set. Proceeding in this way we obtain a finite number of independent sets, say , $V_1, V_2, \ldots, V_s$.

Now we claim that for $t < s$

(i) $V_t \cap V_s = \phi$

(ii) $| V_t | \geq |V_s|$.

(i) Let $t < s$. If possible let $V_t \cap V_s \neq \phi$. Then there is a vertex $x \in V_t \cap V_s$, so that $x \in V_t$ and $x \in V_s$. Then $x = t + pd_0$, $x = s + qd_0$ for some positive integers p and q. Since $t \neq s$, we have $p \neq q$.

Let $p > q$. Then $t + pd_0 = s + qd_0$ implies that $t + (p - q) d_0 = s$, where $p - q$ is an integer. So $s = t + (p - q) d_0 \in V_t$ and $p - q < p$. This shows that $s \in V_t$, which is a contradiction to the fact that $s \notin V_t$.

Let $p < q$. Now $s = s + 0d_0 \in V_s$ shows that s is the smallest number in V_s and $s > t$. Again $t + pd_0 = s + qd_0$ implies that $t = s + (q - p)d_0 \in V_s$, where $q - p < q$ is an integer. This shows that $t \in V_s$. This is a contradiction to the fact that s is the smallest number in V_s and $s > t$. So, $t \notin V_s$. These show that, there is no common elements between V_t and V_s so that $V_t \cap V_s = \phi$.

(ii) Let $| V_t | = k_1$ and $| V_s | = k_2$. Then

$V_t = \{ t, t + d_0 , t + 2d_0 , \ldots, t + (k_1 - 1) d_0 \}$, where k_1 is the least positive integer such that $t + (k_1 - 1)d_0 < n$. Similarly

$V_s = \{ s, s + d_0 , s + 2d_0, \ldots, s + (k_2 - 1)d_0 \}$, where k_2 is the least positive integer such that $s + (k_2 - 1)d_0 < n$. Since $t < s$ and $t + rd_0 < s + rd_0$. This together with $t + (k_1 - 1) d_0 < n$ and $s + (k_2-1)d_0 < n$ implies that $k_1 \geq k_2$.

Remark : 3.4 :

The following is the procedure for finding the independent sets and the independence number of $G(Z_n, D)$.

Consider the graph $G(Z_n, D)$. Let d_0 be the smallest positive integer that does not divide n. We start with vertex 1 and find the corresponding independent set, $V_1 = \{ 1 + rd_0 / 0 \leq r \leq k - 1$, where k is the least positive integer such that $rd_0 \notin S^*$ and $1 + rd_0 < n \}$.

Next let t be least positive integer such that $t \notin V_1$. Consider the corresponding independent set, $V_t =\{ t + rd_0 / 0 \leq r \leq k - 1$,where k is the least positive integer such that $rd_0 \notin S^*$ and $t + rd_0 < n$ }.

Proceeding in this way, we get a finite number of independent sets, say, V_1, V_2, …., V_s which are such that

(i) $V_t \cap V_s = \phi$ if $t \neq s$

(ii) $|V_1| \geq |V_2| \geq ….. \geq |V_s|$.

So, V_1 is one of the largest independent set of $G(Z_n , D)$ so that $|V_1|$ is the independence number of $G(Z_n , D)$.

This discussion leads to the following corollaries, which give the independence number and the chromatic number of $G(Z_n, D)$.

Corollary: 3.5:

Let n = 5, or, $n \geq 7$ and d_0 be the smallest positive integer that does not divide n. Then the independence number $\alpha(G(Z_n, D)) $ is given by $\alpha(G(Z_n, D)) = k$, where k is the least positive integer such that $1 + kd_0 < n$ and either kd_0 divides n, or, $(n – k d_0)$ divides n.

Proof :

Let n = 5, or, $n \geq 7$ and d_0 be the smallest positive integer that does not divide n.

$V_1 = \{ 1 + rd_0 / 0 \leq r \leq k – 1$, where k is the least positive integer such that $rd_0 \notin S^*$ and $1 + rd_0 < n$ }.

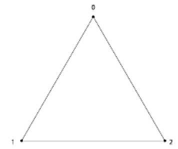
From this, we see that k is the least positive integer such that $kd_0 \in S^*$. That is, k is the least positive integer such that either kd_0 divides n or $(n – kd_0)$ divides n and this k is the independence number of $G(Z_n, D)$.

Remark: 3.6

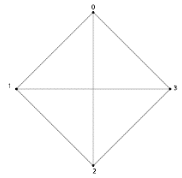
For n = 2, 3, 4, 6, the independence number of $G(Z_n, D)$ is 1 since $G(Z_n, D)$ is a complete graph.

Example: 3.7:

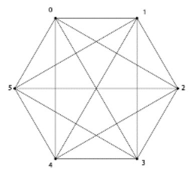
$G(Z_2, D)$



$G(Z_3, D)$



$G(Z_4, D)$



$G(Z_6, D)$

For n = 2, 3, 4 and 6, the independence number of $G(Z_n, D)$ is 1, since $G(Z_n, D)$ is a complete graph.

Definition: 3.8:

1. **A k – vertex colouring** of a graph G is an assignment of k – colours , 1, 2, .., k, to the vertices of G, So, a k – vertex colouring of G partitions V into the partition (V_1, V_2,…, V_k), where V_i is the subset of vertices of V which are coloured by i.
2. The colouring of G is **proper** if no two adjacent vertices have the same colour.
3. Thus a **proper k – vertex colouring** of a loopless graph G is a partition (V_1, V_2,…, V_k) of the vertex set V of G into k independent sets.
4. G is called **k-colourable** if G has a proper k-vertex colouring.

5. The **chromatic number,** $\chi(G)$, of G is the minimum k for which G is k-colourable. If $\chi(G) = k$, G is said to be **k-chromatic.**

Corollary: 3.9:

For n = 5, or, $n \geq 7$ the Chromatic number of divisor Cayley graph of $G(Z_n, D)$ is s, where s is the number of disjoint independent subsets of $G(Z_n, D)$.

Proof:

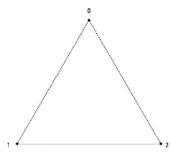
Proof is immediate from the Remark 3.4

Remark: 3.10:

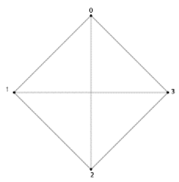
For n = 2, 3, 4, 6, the chromatic number of $G(Z_n, D)$ is n, since $G(Z_n, D)$ is a complete graph.

Example: 3.11:

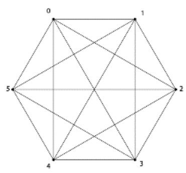
G(Z2, D)



G(Z3, D)



G(Z4, D)



G(Z6, D)

The chromatic number of $G(Z_2, D)$, $G(Z_3, D)$, $G(Z_4, D)$ and $G(Z_6, D)$ are 2, 3, 4 and 6.

4. Conclusion

Certain domination parameters of the Divisor Cayley graph relating to independent set, independence number and chromatic number have been studied. Domination parameters like total domination number, bondage number of this graph are under study.

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