A STUDY ON VERTEX DOMINATION IN UNITARY DIVISOR CAYLEY GRAPHS

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Abstract:
Let $n \geq 1$ be an integer and let S be the set of unitary divisors of n, other than n. Then the set $S^* = \{s, n - s / s \in S\}$ is a symmetric subset of the group $(Z_n, \oplus)$, the additive abelian group of integers modulo n. The Cayley graph of $(Z_n, \oplus)$) associated with the above symmetric subset S^* is called the ***unitary divisor Cayley graph*** *and it is denoted by $G(Z_n, U_n)$. That is, the graph $G(Z_n, U_n)$ is the graph whose vertex set is $V = Z_n$ and the edge set E is the set of all ordered pairs of vertices x, y such that either $x - y \in S^*$, or, $y - x \in S^*$. In a graph G, a vertex v and an edge e in G are said to cover each other if they are incident. A set S of vertices which covers all the edges of a graph G is called vertex cover of G, in the sense that every edge of G is incident with some vertex in S. A minimum vertex cover is the one with minimum cardinality. In this paper we study the vertex cover and vertex covering number of the unitary divisor Cayley graph $G(Z_n, U_n)$*

Key words: Unitary divisor, Unitary divisor Cayley graph, Vertex Set, Vertex cover, Vertex covering number.

1. Introduction

Nathanson [9] was the pioneer in introducing the concepts of number theory into graph theory and thus paved way for the study of a new class of graphs called Arithmetic Graphs arising by defining adjacency using various arithmetic functions. The theory of groups provides an interesting and powerful abstract approach to the study of symmetries of various graphs.

A new class of graphs namely, Cayley Graphs can be constructed by making use of a group X and a symmetric subset S of X (a subset S of X is called a symmetric subset if $s \in S \Rightarrow s^{-1} \in S$). It is the graph $G(X, S)$, whose vertex set is X and edge set $E = \{(x, y)/x, y \in X \text{ and either } xy^{-1} \in S, \text{ or}, yx^{-1} \in S\}$. It is well known that [Th. 1.4.5, p 16 of 8] $G(X, S)$ is an undirected graph without loops, which is $|S^*|$ − regular having $\frac{|X||S|}{2}$ edges. The cycle structure of Cayley graphs and Unitary Cayley graphs were studied by Berrizbeitia and Guidicci [2,3] and Detzer and Guidicci [5]. Madhavi [8] studied Arithmetic Cayley graphs associated with quadratic residues modulo p, a prime, the Euler-Totient function $\varphi(n)$ and the divisor function $d(n)$, $n \geq 1$ an integer.

For standard terminology and notions in graph theory, we refer Bondy and Murty [4] and Harary [7] and for number theoretic notions Apostol [1] and Eckford Cohen [6] .

2. Unitary Divisor Cayley Graph and Properties

Let $n \geq 1$ be an integer. Consider the set $Z_n = \{\overline{0}, \overline{1}, \dots, \overline{n-1}\}$ of residue classes modulo n. Since $\overline{n} = \overline{0}$, we can as well denote $Z_n = \{\overline{1}, \overline{2}, \dots, \overline{n}\}$. In view of this, the set Z_n is henceforth represented by $Z_n = \{\overline{1}, \overline{2}, \dots, \overline{n}\}$, or, simply $Z_n = \{1,2, \dots, n\}$. In the abelian group $(Z_n, \oplus)$, n is the identity element and $n - r$ is the inverse of r in $(Z_n, \oplus)$.

Definition 2.1: Let $n \geq 1$ be an integer. A divisor d of n which is such that $(n, \frac{d}{n}) = 1$ is called a **unitary divisor** of n. The number of unitary divisors of n is denoted by $u(n)$ and the set of unitary divisors of n is denoted by U_n.

For example, for $n = 8$, the unitary

divisors are 1,8 and $u(n) = 2$, while for $n = 120$, the unitary divisors are 1,3,5,8,15,24,40,120 and $u(n) = 8$.

In the following table the unitary divisors and their number are given for integers n up to $n = 15$.

$\boldsymbol{n}$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Unitary divisors of $\boldsymbol{n}$	1	1, 2	1, 3	1, 4	1, 5	1, 2, 3,	1, 7	1, 8	1, 9	1, 2, 5, 10	1, 11	1, 3, 4, 12	1, 13	1, 2, 7, 14	1, 3, 5, 15
$\boldsymbol{u(n)}$	1	2	2	2	2	4	2	2	2	4	2	4	2	4	4

Let S be the set of unitary divisors of n, **other than $\boldsymbol{n}$**. The set S need not be a symmetric subset of the group $(Z_n, \oplus)$. For example, for $n = 12$, the set S of unitary divisors of n, other than n is $S = \{1,3,4\}$. Now for $3 \in S$, its inverse in $(Z_{12}, \oplus)$ is $n - 3 = 12 - 3 = 9$, which is not a unitary divisor of 12. However, the set $S^* = \{u, n - u \,/\, u \textit{ is a unitary divisor of } n\}$ is a symmetric subset of the group $(Z_n, \oplus)$. Using this symmetric subset of $(Z_n, \oplus)$, the unitary divisor Cayley graph is defined as follows:

Definition 2.2: Let $n \geq 1$ be an integer and let $S^* = \{u, n - u \,/\, u \textit{ is a unitary divisor of } n \textit{ other than } n\}$. The **Unitary Divisor Cayley graph** $G(Z_n, U_n)$ is the graph, whose vertex set is $V = \{1, 2, \dots, n-1, n\}$ and the edge set $E = \{(x, y) / \, x - y \in S^*, \, or, \, y - x \in S^*\}$.
For $n = 4, 6, 8, 12$ the unitary divisor Cayley graphs are given below:

Fig: $G(Z_4, U_4)$

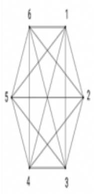

Fig: $G(Z_6, U_6)$

Fig: $G(Z_8, U_8)$

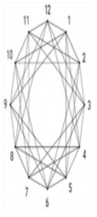

Fig: $G(Z_{12}, U_{12})$

(i) If $n = 2k$, where k is odd, then $|S^*|$ is odd.
(ii) If $n = 2^r q$, where $r > 1$ and q is odd, then $|S^*|$ is even.
(iii) If n is odd, then $|S^*|$ is even.
(iv) The graph $G(Z_n, U_n)$ is $|S^*|$ - regular. Moreover the number of edges in $G(Z_n, U_n)$ is $\dfrac{n\,|S^*|}{2}$
(v) The graph $G(Z_n, U_n)$is connected.
(vi) The graph $G(Z_n, U_n)$ is Hamiltonian
(vii) The cycle $(1, 2, 3, \dots, (n-1), n, 1)$ is called the outer Hamilton Cycle of the graph $G(Z_n, U_n)$.

3. Main Results

Definition 3.1:

A vertex v and an edge e are said to **cover** each other if they are incident in G. A set S of vertices which covers all the edges of a graph G is called the **vertex cover** of G, in the sense that every edge of G is incident with some vertex in S.

A minimum vertex cover is the one with minimum cardinality.

Definition: 3.2:

The cardinality of a minimum vertex cover of a graph G is called the **covering number** and is denoted by β(G).

Remark: 3.3:

The vertex covering number is n – 1, for n = 2, 3, 6, since $G(Z_n, U_n)$ is a complete graph for n = 2, 3, 6.

Theorem: 3.4:

Let n ≥ 4 be a power of a prime. Then the minimum vertex cover of $G(Z_n, U_n)$ is $\{1, 3, 5, \dots, p\}$, where p is an odd integer $\leq$ n.

Proof:

Let us consider the unitary divisor Cayley graph $G(Z_n, U_n)$.

Suppose that $n \geq 4$ is a power of a prime. Then 1 is the only unitary divisor of n other than n so that $S^* = \{1, n - 1\}$. Let $V_1 = \{1, 3, 5, \dots, p\}$, where p is an odd integer ≤ n. Let n be an edge of $G(Z_n, U_n)$. Then e = (r, s), where $1 \le r, s \le n$ and either r – s, or, s – r is in S^*.

We claim that one of r and s is in V_1. For, if $r \notin V_1$ and $s \notin V_1$, then r = 2α and s = 2β , where α, β are positive integers so that r – s = 2(α - β). If n is even, then 2α ≤ n, 2β ≤ n, so that $\alpha \le \frac{n}{2}$, $\beta \le \frac{n}{2}$. Now $\alpha - \beta < \frac{n}{2}$ implies 2(α - β) < n. 2(α - β) < n – 1. So $2(\alpha - \beta) \notin S^*$. If n is odd, then n – 1 is even. Also 2α ≤ n – 1, 2β ≤ n – 1 so that $\alpha \le \frac{n-1}{2}$, $\beta \le \frac{n-1}{2}$ and $\alpha - \beta < \frac{n-1}{2}$, or, 2(α - β) < n – 1.So $2(\alpha - \beta) \notin S^*$. Hence $r - s \notin S^*$. Similarly, one can show that $s - r \notin S^*$. This is a contradiction. So, at least one of r and s must belong to V_1, which implies that V_1 is a vertex cover of $G(Z_n, U_n)$.

Let us now show that V_1 is the minimal vertex cover of $G(Z_n, U_n)$. For this, consider the set $V_1 - \{i\}$, for any $i \in V_1$. Then i being odd, i = 2t – 1, for some positive integer t. Now the edge (2t – 1, 2t), (this is an edge, since $2t - (2t - 1) = 1 \in S^*$), is not covered by the set $V_1 - \{i\}$, since 2t being even, $2t \notin V_1 - \{i\}$ and $2t - 1 = i \notin V_1 - \{i\}$. So, V_1 is the minimum vertex cover of $G(Z_n, U_n)$.

The following corollary is immediate from the Theorem.

Corollary: 3.5:

If n ≥ 4 is a power of a prime, then the vertex covering number $\beta(G(Z_n, U_n)) = \left\lceil \frac{n+1}{2} \right\rceil$.

Example: 3.6: The minimum vertex cover of $G(Z_8, U_8)$ is {1,3,5,7} and vertex covering number $\beta(G(Z_8, U_8)) = \left\lceil \frac{8+1}{2} \right\rceil = 4$.

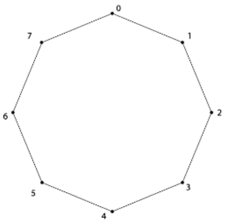

Fig: $G(Z_8, U_8)$

Theorem: 3.7:

Let n be not power of a prime and $n \ne 2^r .3^s$, r, s are integers ≥ 1. Let d_0 be the smallest positive integer that does not divide n.

(i) The set $V_0 = V - U_0$, where $U_0 = \{kd_0 / 1 \le k \le r$ such that $rd_0 \in S^*$ and for t < r, $td_0 \notin S^*\}$ is a vertex cover of $G(Z_n, U_n)$.

(ii) For any positive integer $d > d_0$ that does not divide n, let $U_d = \{u / u = d + kd_0$, if u ≤ n and $u = d + kd_0 - n$, if u > n, $0 \le k \le r - 1\}$, then the set $V_d = V - U_d$ is also a vertex cover of $G(Z_n, U_n)$.

(iii) $|V_{d_0}| = |V_d|$

Proof:

Let us consider the unitary divisor Cayley graph $G(Z_n, U_n)$.

(i). Suppose that n is not power of a prime and $n \ne 2^r .3^s$, where r, s are integers ≥ 1. Let d_0 be the smallest positive integer that does not divide n. Let $V_0 = V - U_0$, where $U_0 = \{kd_0 / 1 \le k \le r$ such that $rd_0 \in S^*$ and for t < r, $td_0 \notin S^*\}$. Let e be an edge of $G(Z_n, U_n)$. Then e = (i, j), where 1 ≤ i, j ≤ n and either i – j, or, j – i is in S^*.

We claim that one of i and j is in V_0. For, if $i \notin V_0$ and $j \notin V_0$, then $i = sd_0$ and $j = td_0$, 1 ≤ s, t ≤ r. Suppose i > j, then $i - j = sd_0 - td_0 = (s - t)d_0 < rd_0$. So, $(s - t)d_0 \notin S^*$. Hence $i - j \notin S^*$. Similarly, if i < j, then $j - i \notin S^*$. This shows e = (i, j) is not an edge of $G(Z_n, U_n)$, which e = (i, j)is not an edge of $G(Z_n, U_n)$, which is a contradiction. So, at least one of i and j must belong to V_0, which implies that V_0 is a vertex cover of $G(Z_n, U_n)$.

(ii). Let d be the positive integer that does not divide n. Let $V_d = V - U_d$, where $U_d = \{u / u = d + kd_0$, if u ≤ n and u = d +

$kd_0 - n$, if $u > n$, $0 \le k \le r - 1\}$. Let e be an edge of $G(Z_n, U_n)$. then $e = (i, j)$, where $1 \le i, j \le n$ and either $i - j$, or, $j - i$ is in V_d. Suppose $i \notin V_d$ and $j \notin V_d$. Here three cases will arise.

Case i:

Let $i, j \le n$. Then $i = d + sd_0$, $j = d + td_0$, for some s and t, $0 \le s, t \le r - 1$.
Now $i - j = (d + sd_0) - (d + td_0) = sd_0 - td_0 = (s-t)d_0 \notin S^*$, since $s - t < r$. Similarly $i < j$, $j - i = (d + td_0) - (d + sd_0) = td_0 - sd_0 = (t - s)d_0 \notin S^*$, since $t - s < r$. This shows $e = (i, j)$ is not an edge of $G(Z_n, U_n)$, which is a contradiction. So, atleast one of i and j must belong to V_d, which implies that V_d is a vertex cover of $G(Z_n, U_n)$.

Case ii:

Let $i, j > n$ and $i > j$. Then $i = (d + sd_0) - n$, $j = (d + td_0) - n$, for some s and t, $0 \le t < s \le r - 1$. Now $i - j = [(d + sd_0) - n] - [(d + td_0) - n] = sd_0 - td_0 = (s - t)d_0 \notin S^*$, since $s - t < r$. This shows $e = (i, j)$ is not an edge of $G(Z_n, U_n)$, which is a contradiction. So, atleast one of i and j must belong to V_d, which implies that V_d is a vertex cover of $G(Z_n, U_n)$.

Case iii:

Let $i \le n$ and $j > n$. Then $i = d + sd_0$ and $j = d + td_0 - n$, $t > s$. Now $i - j = (d + sd_0) - (d + td_0 - n) = n - (t-s)d_0$. We claim that $n - (t-s)d_0 \notin S^*$ For, if $n - (t - s)d_0 \in S^*$. Then $n - [n - (t - s)d_0] \in S^*$ or, $(t - s)d_0 \in S^*$, a contradiction since $t - s < r$. So, $i - j \notin S^*$. This shows $e = (i, j)$ is not an edge of $G(Z_n, U_n)$, which is a contradiction. So, atleast one of i and j must belong to V_d which implies that V_d is a vertex cover of $G(Z_n, U_n)$.

(iii) . Since $|U_{d_0}| = |U_d|$, we have $|V_{d_0}| = |V_d|$.

The following corollary is immediate from the Theorem.

Corollary: 3.8:

Let n be not power of a prime and $n \neq 2^r . 3^s$, where r, s are integers ≥ 1 and let d_0 be the smallest positive integer that does not divide n. Then the vertex covering number of $G(Z_n,U_n)$is given by $\beta(G(Z_n, U_n)) = n - l$, where $l = |U_{d_0}|$ and

$U_{d_0} = \{kd_0 / 1 \le k \le r$ such that $rd_0 \in S^*$ and for $t < r$, $td_0 \notin S^*\}$.

Proof:

From part (ii) of Theorem 3.7, we conclude that for every integer $d > 0$, which is not a divisor of n, we get a vertex cover V_d, namely,

$V_d = V - U_d$ where

$U_d = \{u / u = d + kd_0$ if $u \le n$ and $u = d + kd_0 - n$ if $u > n$, $0 \le k \le r - 1\}$
and d_0 is the smallest positive integer that does not divide n.

Also, from part (iii) of the Theorem 3.7, we get $|V_{d_0}| = |V_d|$. So $|V_{d_0}|$ is the vertex covering number of $G(Z_n, U_n)$,which is $n - l$, where $l = |U_{d_0}|$.

Remark: 3.9:

The vertex cover and vertex covering number of unitary divisor Cayley graph $G(Z_n, U_n)$, where $n = 2^r . 3^s$, $r \ge 1$, $s \ge 1$ integer are not covered in this paper. They need further study.

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