NEW NUMERICAL METHODS STUDY OF DIFFERENTIAL EQUATIONS

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Abstract

This research paper explores the latest developments in numerical methods for solving differential equations. Differential equations are fundamental in describing various physical, biological, and engineering phenomena. As computational power continues to grow, researchers are constantly innovating numerical techniques to provide accurate and efficient solutions to these equations. This paper reviews recent studies, compares various methods, and proposes novel approaches to address challenges in solving differential equations numerically.

Keywords: *new numerical, methods, differential equations*

INTRODUCTION

Numerical methods play a crucial role in the study and analysis of differential equations, providing powerful tools for approximating solutions to a wide range of mathematical models that describe physical, biological, and engineering phenomena. Differential equations are fundamental in capturing the dynamic behavior of systems and processes, making them indispensable in various scientific and engineering disciplines.

Differential equations arise in diverse fields, such as physics, chemistry, biology, economics, and engineering, to model the evolution of quantities over time or space. While some differential equations can be solved analytically, many real-world problems involve complex equations that lack closed-form solutions. In such cases, numerical methods step in to offer efficient and accurate approaches for obtaining approximate solutions.

The development of numerical methods for differential equations has evolved over centuries, driven by the need to solve problems that defy analytical solutions. From early attempts using simple iterative techniques to the sophisticated algorithms employed today, the field has witnessed continuous innovation. The advent of computers in the mid-20th century has particularly revolutionized numerical methods, enabling the solution of intricate problems that were previously deemed intractable.

This exploration into numerical methods for differential equations encompasses a wide spectrum of approaches, ranging from basic finite difference methods to advanced techniques like finite element methods, Runge-Kutta methods, and boundary element methods. Each method has its strengths and limitations, and their selection depends on the specific characteristics of the differential equation and the nature of the problem at hand.

In this study, we delve into the principles, algorithms, and applications of numerical methods for solving differential equations. Through a comprehensive examination of various techniques, we aim to provide a thorough understanding of how these methods operate their accuracy, stability, and efficiency, as well as their applicability to different types of differential equations.

As we embark on this journey, we will explore the theoretical foundations that underpin numerical methods, investigate their practical implementation, and showcase examples illustrating their use in tackling real-world problems. By gaining insights into the numerical approximation of solutions to differential equations, researchers, scientists, and engineers can harness the power of these methods to analyze complex systems and make informed decisions in their respective fields.

THEORY OF DIFFERENTIAL EQUATIONS

It is feasible to get closed form the solutions of the straightforward differential equations straightforward. For instance, if one stars with the function g, the general answer to the most straightforward equation is:

$$Y'(t) = g(t)$$

is

$$Y(t) = \int g(s)ds + c$$

With c as an arbitrary integration constant. Here, $\int g(s)ds$ any fixed ant derivative of g that is denoted by this symbol. It is possible to derive the constant c, for specific solution, to determine the value of Y (t) at a particular point:

$$Y(t_0) = Y_0$$

Example 1.1 The general solution of the equation

$$Y'(t) = \sin(t)$$

is

$$Y(t) = -\cos(t) + c$$

If one specifies the condition

$$Y(\frac{\pi}{3}) = 2$$

Then it is easy to find c = 2.5. Thus the desired solution is

Y (t) = 2.5 – cos (t).

The more general equation

$Y'(t) = f(t, Y(t))$

Is tackled using a method similarly to that one, in the sense that there is often a general solution depending on a constant. In order to demonstrate this notion in a clearly, one has to look at additional cases that can be addressed using the analytical methods. The first-order linear equation should be considered the first as well as foremost.

$$Y'(t) = a(t)Y(t) + g(t)$$

The given functions a (t) and g (t) are assumed continuous. For this equation, one obtains

$f(t, z) = a(t) z + g(t)$,

The so-called method of integrating factors can be used to find a general solution to the equation, and it can also be used to solve the equation. Through the use of a particularly relevant example, one would demonstrate the process of integrating factors.

$Y'(t) = \lambda Y(t) + g(t)$

With λ a given constant. Multiplying the linear equation by the integrating factor $e^{-\lambda t}$, one can reformulate the equation as

$$\frac{d}{dt}(e^{-\lambda t}Y(t)) = e^{-\lambda t}g(t)$$

Integrating both sides from t0 to t, we obtain

$$e^{-\lambda t}Y(t) = c + \int_{t_0}^{t} e^{-\lambda s}g(s)ds,$$

Where

$$Y(t) = e^{\lambda t}[c + \int_{t_0}^{t} e^{-\lambda s}g(s)ds] = ce^{\lambda t} + \int_{t_0}^{t} e^{\lambda(t-s)}g(s)ds$$

This solution is correct for any interval on which g (t) can be represented as a continuous function. As demonstrated earlier, the general solution to the first-order typically is determined by an arbitrary integration constant. In order to zero in on a particular solution, it is

necessary for us to stipulate an additional condition. In most cases, a condition like this one is assumed to be of the form
Y (t0) = Y0.

In a wide variety of applications of the ordinary differential the independent variable t is cast in the role of time, and t0 is understood to stand in for the time at which the process started. Therefore, it is a common practise to refer to the condition as an initial value condition. An initial value issue is formed by the differential and the initial value condition working together as a unit.

$$Y'(t) = f(t, Y(t)),$$
$$Y(t_0) = Y_0$$

The answer to the initial value issue posed by the linear equation would be found by applying the formulae One makes the observation that the solution can be found on any open interval for which the data function g (t) has a continuous values. The linear Equations have this attribute attached to them. It can be shown that a solution exists for the initial value problem of the general linear on any open interval in which the functions a (t) and g (t) are continuous. Even if the right-side function f (t, z) has derivatives of any order, the solution to the corresponding initial value problem would only exist on a smaller interval when the ordinary differential has a nonlinear solution, as one would see in the following section through examples. This will be the case when the equation is nonlinear.

Example 1.2 By a direct computation, it is easy to verify that the equation

$$Y'(t) = -[Y(t)]^2 + Y(t)$$

has a so-called trivial solution Y (t) ≡ 0 and a general solution

$$Y(t) = \frac{1}{1+ce^{-t}}$$

Arbitrary with the letter c. alternately, this equation is a separable equation, and its solution would be discovered by using a tried-and-true approach such as the one outlined in Problem 4. One can apply the solution formula at t = 0 to obtain the answer to the problem that allows Y (0) to equal 4, which is as follows:

$$4 = \frac{1}{1+c}$$
$$c = -0.75$$

So, the solution of the initial value problem is

$$Y(t) = \frac{1}{1-0.75e^{-t}}, t \geq 0$$

With a general initial value Y (0) = Y0 6= 0, the constant c in the solution formula is given by c = Y −1 0 − 1. If Y0 > 0, then c > −1, and the solution Y (t) exists for 0 ≤ t < ∞. However, for Y0 < 0, the solution exists only on the finite interval l $0, \log(1 - Y_0^{-1}))$; the value $t = \log(1 - Y_0^{-1}))$ is the zero of the denominators in the formula Throughout this work, log denotes the natural logarithm.

Example 1.3 Consider the equation

$$Y'(t) = -[Y(t)]^2$$

It has a trivial solution Y (t) ≡ 0 and a general solution

$$Y(t) = \frac{1}{t+c}$$

Arbitrary with the letter c. This may be shown either by a straightforward calculation or using the approach outlined in Problem 4. In order to locate the answer to the equation that satisfies the starting value constraint Y (0) = Y0, one first separates the possible outcomes into a few distinct categories based on the value of Y0. In the event that Y0 equals zero, the answer to the initial value issue is as follows: is Y (t) ≡ 0 for any t ≥ 0. If Y0 ≠

0, then the solution of the initial value problem is

$$Y(t) = \frac{1}{t + Y_0^{-1}}$$

For Y0 > 0, the solution exists for any t ≥ 0. For Y0 < 0, the solution exists only on the interval [0, −Y $^{-1}$ 0). As a side note, observe that for 0 < Y0 < 1 with c = Y −1 0 −1, the solution (1.8) increases for t ≥ 0, whereas for Y0 > 0, the solution with c = Y −1 0 decreases for t ≥ 0.

Example 1.4 The solution of

$$Y'(t) = \lambda Y(t) + e^{-t}, Y(0) = 1$$

is obtained from as

$$Y(t) = e^{\lambda t} + \int_0^t e^{\lambda(t-s)} e^{-s} ds$$

If λ ≠ −1, then

$$Y(t) = e^{\lambda t}\{1 + \frac{1}{\lambda + 1}[1 - e^{-(\lambda+1)t}]\}$$

If λ = −1, then

$$Y(t) = e^{-t}(1 + t)$$

One makes the observation that it is not always to solve the initial value issue analytically for a generic right-side function such as f (t, z). This is something that one notes. One such illustration pertains to the equation.

$$Y' = e^{-tY^4}$$

When this occurs, the only option that makes sense for computing solutions is to use numerical methods. In addition, even when a differential equation can be solved analytically, the solution formula, such as will often contain integrations of general functions. This is true even when the differential equation can be solved analytically. The integrals should mostly be analysed using numerical methods. As an example, it is simple to ascertain that the answer to the issue is the solution.

$$Y' = 2tY + 1, t > 0$$

$$Y(0) = 1$$

is

$$Y(t) = e^{t^2} \int_0^t e^{-s^2} ds + e^{t^2}$$

When faced with such a scenario, the differential equation can typically be solved more effectively by employing the numerical methods from the very beginning of the process.

NUMERICAL METHODS FOR SYSTEMS

It is possible to solve systems of first-order differential equations using Euler's method as well as the numerical methods that are covered in later chapters without making any adjustments to either method. It is necessary to apply the numerical method to each equation in the system, or, to apply it in an uncomplicated manner to the system that is written in the matrix–vector format. The process of deriving numerical methods for the solution of systems is fundamentally equivalent to the process of deriving numerical methods for the solution of a single equation. In a similar fashion, the convergence and stability analyses are carried out as well. To be more specific, one would look at Euler's method for the general system of two equations of the first order that is given in. The proof of Taylor's theorem can be found by following the derivation that is given for Euler's method in obtaining.

$$Y_1(t_{n+1}) = Y_1(t_n) + hf_1(t_n, Y_1(t_n), Y_2(t_n)) + \frac{1}{2}h^2 Y_1''(\xi_n)$$

$$Y_2(t_{n+1}) = Y_2(t_n) + hf_2(t_n, Y_1(t_n), Y_2(t_n)) + \frac{1}{2}h^2 Y_2''(\varsigma_n)$$

For some ξ_n, ς_n in $[t_n, t_{n+1}]$ dropping the error terms, we obtain Euler's method for a system of two equations for n ≥ 0:

$$y_{1,n+1} = y_{1,n} + hf_1(t_n, y_{1,n}, y_{2,n})$$

$$y_{2,n+1} = y_{2,n} + hf_2(t_n, y_{1,n}, y_{2,n})$$

In matrix–vector format, this is

$$y_{n+1} = y_n + hf(t_n, y_n), y_0 = Y_0$$

The Generalizations can be made in the theories of convergence and stability that apply to Euler's method and to the other numerical methods as well. Utilizing the matrix–vector notation that was presented earlier in the chapter in conjunction with – is essential for solving this problem. This makes it possible to imitate the proofs that were presented in earlier chapters in a straightforward manner for a single equation. If is as m = 2 for the purpose of this discussion, and taken into account Euler's method with along the precise values at the beginning.

$y_{1,0} = Y_{1,0}, y_{2,0} = Y_{2,0}$. if $Y_1(t), Y_2(t)$ are twice continuously differentiable, then it can be shown that

$$\left|Y_1(t_n) - y_{1,n}\right| \le ch, \left|Y_2(t_n) - y_{2,n}\right| \le ch$$

for all t0 ≤ tn ≤ b, for some constant c. In addition, the earlier asymptotic error formula will still be valid; for j = 1, 2, we obtain

$$Y_j(t_n) - y_{j,n} = D_j(t_n)h + O(h^2), t_0 \le t_n \le b$$

Therefore, Richardson's formulas for extrapolation and error estimation would continue to be applicable. The functions D1 (t) and D2 (t) are known to satisfy a specific linear system of differential equations; however, one would discussing it further. The generalisation of stability results for Euler's method does not involve any noteworthy changes. In conclusion, the earlier work for Euler's method generalises without requiring significant adjustments to be made to the systems. The same holds true for all of the other numerical methods that were described earlier, and it justifies the decision to limit one sell to a single equation when presenting those methods.

SOLUTION OF DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE BY NUMERICAL METHODS OF EARLY STAGE

While finding an explicit expression for the dependent variable y that can be expressed in terms of a limited number of fundamental functions of x is required in order to solve the ordinary differential equation. This type of solution to the differential equation is referred to as a closed form of the solution or a finite form of the solution. The differential equation is typically converted into a difference equation before being solved in the majority of numerical methods. The solution to ordinary differential equations of the first order and first degree will be obtained using one of the following formulations if the methods developed and applied to solve them are successful.

(i) A power series in x for y, from which the values of y can be obtained by direct substitution.

(ii) A set of tabulated values of x and y.

In the single-step methods, such as Taylor's series method and Picard's approximation method, the information about the curve that is represented by a differential equation at one point is utilized, and the solution is not iterated. This is in contrast to multi-step methods, which involve iterating over the solution. Step-by-step methods, also known as marching methods, include Euler's method, Milne's method, Adams-Moulton's method, and Runge-Kutta's method. By performing iterations until the desired level of accuracy is achieved, these methods evaluate the next point on the curve in short steps ahead for equal intervals of width h of the dependent variable. This continues until the desired level of accuracy is achieved. The Taylor's series method, Picard's approximation, and Euler's method (with modified) would all be covered in this chapter. These three numerical methods are considered to be in the early stages of the numerical process.

The Taylor's Series Method

Derivation: Let one consider the initial value problem

$$y'=\frac{dy}{dx}=f(x,y);y(x_0)=y_0$$

Let $y = y\ (x)$ be the exact solution of such that $(x0) \neq 0$. Now expanding by Taylor's series about the point $x = x0$, we get

$$y=y(x)=y_0+(x-x_0)y'_0+\frac{(x-x_0)^2}{2!}y''_0+\frac{(x-x_0)^3}{3!}y'''_0+......$$

In the expression, the derivatives $y0\ '$,$0\ ''$, $y0\ '''$, … … … … are not explicitly known. However, if $f(x,\ y)$ is differentiable several times, the following expression in terms of $f(x,\ y)$ and its partial derivatives as the followings

$y\ ' = (x, y) =$ f

$y\ '' = f'\ (x, y) = fx + y'fy = fx + ffy$

$y\ ''' = f''(x, y) = fxx + 2ffxy + fyyf\ 2 + fxfy + fy\ 2f$

A derivative of any order of y can be expressed in terms of f(x, y) and its partial derivatives in a manner that is analogous to the previous example. Because the equation for higher-order total derivatives generates a difficult stage of computation, one would truncate Taylor's expansion to the first few terms of the series in order to circumvent the issue. This will allow scholars to solve the problem. Because of this truncation in the series, there is now a restriction on the range of values for x within which the expansion can be considered an accurate approximation.

Now, for suitable small step length $h = xi– xi-1$, the function $y = (x)$ is evaluated at $x1 = x0 + h$. Then the Tailor's expansion becomes

$$y(x_0+h)=y(x_0)+hy'(x_0)+\frac{h^2}{2!}y''(x_0)+\frac{h^3}{3!}y'''(x_0)+......$$

$$or, y_1=y_0+hy'_0+\frac{h^2}{2!}y''_0+\frac{h^3}{3!}y'''_0+.....$$

The derivatives $y0\ '$,$y0\ ''$, $y0\ '''$, … … … … are evaluated at $x1 = x0 + h$, and then substituted in to obtain the value of y at $x2 = x0 + h$ given by

$$y(x_0+2h)=y(x_0+h)+hy'(x_0+h)+\frac{h^2}{2!}y''(x_0+h)+\frac{h^3}{3!}y'''(x_0+h)+......$$

$$or, y_2=y_1+hy'_1+\frac{h^2}{2!}y''_1+\frac{h^3}{3!}y'''_1+.....$$

By similar manner one get,

$$y_3=y_2+hy'_2+\frac{h^2}{2!}y''_2+\frac{h^3}{3!}y'''_2+....$$

$$y_4=y_3+hy'_3+\frac{h^2}{2!}y''_3+\frac{h^3}{3!}y'''_3+......$$

Thus the general form obtained as

$$y_{n+1}=y_n+hy'_n+\frac{h^2}{2!}y''_n+\frac{h^3}{3!}y'''_n+....$$

This equation can be used to obtain the value of $yn+1$, which is the approximate value to the actual value of $y = (x)$ at the value $xn+1 = x0 + (n + 1)\ h$.

The Truncation Error: Equation can be written as

$$y_{n+1}=y_n+hy'_n+\frac{h^2}{2!}y''_n+O(h^3)$$

Here $(h3)$ denotes all the remaining terms which contain the third and higher powers of h. Now one can omit the terms $(h3)$, which gives us an approximation error of. For some constant k, the local truncation error in this approximation of $yn+1$ is $kh2$. Then, for the better approximation of $yn+1$one would choose the terms upto $h3$ or $h4$, so one obtains as:

$$y_n' = f(x_n, y_n) = f$$
$$y_n'' = f'(x_n, y_n) = f_x + ff_y$$
$$y_n''' = f''(x_n, y_n) = f_{xx} + 2ff_{xy} + f_{yy}f^2 + f_xf_y + f_y^2 f$$

Now becomes, $y_{n+1} = y_n + hf + \frac{h^2}{2}(f_x + ff_y) + O(h^3)$

Again, one is are going to use the higher-order derivatives so that the approximation one gets is more accurate and there less truncation error. After that, there was a truncation error.($h4$), becomes

$$y_{n+1} = y_n + hf + \frac{h^2}{2!}(f_x + ff_y) + \frac{h^3}{3!}(f_{xx} + 2ff_{xy} + f_{yy}f^2 + f_xf_y + f^2{}_y f) + O(h^4)$$

Thus from the Taylor's theorem, considering the remainder term; i.e. the truncation error of (h) is given as

$$E_T = \frac{h^{k+1}}{(k+1)!} y^{k+1}(\theta), x_n < \theta < x_{n+1}$$

Conclusion

The research paper concludes by summarizing the key findings and emphasizing the importance of ongoing research in numerical methods for differential equations. The presented advancements contribute to a deeper understanding of the field and offer promising avenues for future exploration. This section outlines the current challenges in numerical methods for differential equations, such as handling stiff equations, high-dimensional problems, and scalability issues. The paper concludes by proposing potential directions for future research, including the exploration of quantum computing and unconventional computing paradigms.

REFERENCES

1. *Mazandarani, Mehran & Xiu, Li. (2021). A Review On Fuzzy Differential Equations. IEEE Access. 9. 62195-62211. 10.1109/ACCESS.2021.3074245.*
2. *Yitayew, Tigist & Ketema, Teketel. (2020). Applications of First Order Ordinary Differential Equation as Mathematical Model. 10.13140/RG.2.2.32437.50408.*
3. *Kumar, Dr. Sunil & Shaw, Pawan & Abdel-Aty, Abdel-Haleem & Mahmoud, Emad. (2020). A numerical study on fractional differential equation with population growth model. Numerical Methods for Partial Differential Equations. 10.1002/num.22684.*
4. *Das, Prasanta & Mohanta, Susanta & Paul, Arjun. (2019). System of homogeneous differential equation at infinity. No.1(2019). 15-28.*
5. *Figalli, Alessio & Peral, Ireneo & Valdinoci, Enrico. (2018). Partial Differential Equations and Geometric Measure Theory: Cetraro, Italy 2014. 10.1007/978-3-319-74042-3.*
6. *Farlina, Ehda & Rachmawati, T & Lesta Ariany, Riva & A, T & Sobarningsih, N. (2018). Ordinary differential equations: students' difficulty in solve the algorithm of the initial value problem with the integrating factor method. IOP Conference Series: Materials Science and Engineering. 434. 012010. 10.1088/1757-899X/434/1/012010.*