APPLICATIONS OF FIXED POINTS TO THE SOLUTION OF LINEAR AND NONLINEAR INTEGRAL EQUATIONS

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Abstract

Fixed point theory has become a powerful tool in the analysis and solution of both linear and nonlinear integral equations. The fundamental idea revolves around determining a point in a given space where a mapping (or operator) stabilizes, i.e., the point maps to itself. This approach finds extensive applications in solving integral equations, where the unknown function is often expressed in terms of itself through an integral operator. For linear integral equations, fixed point methods such as Banach's Fixed Point Theorem are utilized to establish the existence and uniqueness of solutions under appropriate conditions. In the case of nonlinear integral equations, more generalized fixed point principles, including Schauder and Krasnoselskii's theorems, offer powerful techniques for proving the existence of solutions, even in the absence of linearity. The application of fixed point theory to these equations not only guarantees the existence and uniqueness of solutions but also provides efficient iterative methods for their numerical approximation. This abstract aims to highlight the theoretical and computational significance of fixed points in addressing challenges associated with linear and nonlinear integral equations, showcasing their role in diverse fields such as physics, engineering, and applied mathematics.

Keywords: *applications, fixed points, solution, linear and nonlinear integral equations*

Introduction

A fixed point, as the name implies, is a point on a curve that remains constant regardless of the transformations applied to it. The history of fixed-point theory may be traced back to the development of topology, a significant branch of mathematics, by Johann Benedict Listing towards the close of the nineteenth century. However, at that time, it was just a concept and not a fully formed field of study. During the early 20th century, the concept of topological spaces was introduced. Afterwards, the fixed-point theory was established by French mathematician H. Poincare. Unlike other areas of mathematics, it was not found until much later, although it is now completely mature. Numerous branches of mathematics are included in the study of fixed point life. Classical and functional analysis are both necessary in analysis, general and algebraic topology are both necessary in topology, and knowledge of operator theory may be used to the study of fixed point existence.

A number of mathematical disciplines have relied on fixed points and fixed point theorems to provide a theoretical framework for their work. These include mathematical economics, approximation theory, game theory, mathematical boundary value problems, initial value problems, and theoretical explanations in many others. Throughout mathematics' history, fixed-point theory has consistently shown its significance. Fixed points have seen a meteoric rise in use thanks to new methods that allow for their discovery with greater efficiency and accuracy. It is now utilized almost everywhere and is no longer restricted to the aforementioned areas of mathematics. Linear, differential, integral, and non-linear integral equations, as well as their fixed points, may be solved and located with the help of fixed point theory. As per the fixed-point theory, it may be said that each maps $T: Y \rightarrow Y$ guarantees either one fixed point or more than one fixed point, and that this map is a self-mapping on Y of a topological space. We may further define such locations as y in Y, where $y = T(y)$. Now that we have established what a fixed point is, let's look at a few real-world instances.

- If an is not equal to zero, then we can translate any mapping $T(y) = y + a$. Because the criteria for a map to have a fixed point are that for any $y, T(y) \neq y$, for $a \neq 0$, this mapping will never have a fixed point.
- Rotation—When a plane is rotated, there is always going to be one fixed point—the place where the plane's center of rotation is located.
- There will always be just two fixed points in the mapping $T(y) = y^2$ specified on $\mathbb{R} \rightarrow \mathbb{R}$. This curve has a fixed point of zero because, at $y = 0$, $y^2 = 0$, or $T(y) = y$. The fixed point of this curve is similarly 1, since for $y = 1$, $y^2 = 1$. That is, $T(y) = y$. This curve will only have two fixed points, namely 0 and 1, since there is no other point for which $T(y) = y$.
- A mapping from $\mathbb{R} \rightarrow \mathbb{R}$ denoted as $T(y) = y^3$, will also always include only three fixed points. Since $T(y) = y$, the fixed point of this curve is 0, since $y = 0$, $y^3 = 0$. Additionally, the fixed point of this curve are 1 and -1 as, at $y = 1$, $y^3 = 1$ and at $y = -1, y^3 = -1$ with $T(y) = y$. This curve will only have three fixed points, 0, 1, and -1, since there is no other point for which T(y) = y. Additionally, for any value of y, T(y) will always be equal to y for mapping $T: Y \rightarrow Y$ defined on $\mathbb{R} \rightarrow \mathbb{R}$. the number of fixed points in this kind of mapping is consequently limitless.

So far, we have only been successful in locating a maximum of three finite fixed locations.

Dutch mathematician L.E.J. Brouwer initially proposed the fixed point theorem in 1912. A theorem grounded in algebraic topology underpinned it. Brouwer demonstrated in his theorem of fixed points that any continuous self-map, T, of any closed ball, Rn, with one unit dimension, would inevitably have a fixed point, y, in Y, such that T(y) = y. Following the validation of Brouwer's theorem, two additional seminal theorems, the "Banach fixed point theorems" and the "Banach contraction principle," were established. A unique fixed point exists in entire metric space if and only if the self-mapping is also a contraction mapping, as shown in this theorem. Therefore, the existence and uniqueness of a fixed point may be simply determined using the Banach contraction principle. In the years after the discovery of the Banach contraction principle, several other mathematicians continued to refine and expand upon it. As a consequence, they were able to solve a wide variety of integral equations, including linear, differential, non-linear, and linear systems. Many new applications of fixed-point theory have their roots in the Banach contraction principle.

REVIEW OF LITERATURE

Ennassik, Mohamed & Lahcen, Maniar & Taoudi, Mohamed. (2021) Several novel fixed point theorems in r-normed and locally r-convex spaces are proven in this study. In addition to partially answering Schauder's hypothesis in the positive, our findings extend other famous discoveries. We establish the equivalent of a Von Neumann's theorem in locally r-convex spaces using the findings that were obtained. Moreover, a game-theoretic application is offered. The writers would like to express their gratitude to the editor and the anonymous reviewers for their insightful criticisms and recommendations that helped refine the first draft of this work.

Asaduzzaman, Md. (2021) We review some important game-theoretic uses of Kakutani's fixed point theorem in this article. Mathematical sets, functions, topological features, and Brouwer's fixed point theorem are the main tools we use in our study to highlight Kakutani's fixed point theorem. New fixed point findings and their applications in many disciplines of game theory may be effectively highlighted by this theory, which is included in the core point of thought.

Hendtlass, Matthew. (2016) We examine Kakutani's expansion of the Brouwer fixed point theorem in this article using Bishop's constructive mathematics as a framework. The fixed point theorem of Kakutani and Brouwer are classically equivalent. Since a finite combinatorial argument is used in the constructive demonstration of (an approximation) Brouwer's fixed point theorem, we are compelled to focus on functions that are uniformly continuous.

Park, Sehie. (2003) The Fan-Browder fixed point theorem, the Kakutani fixed point theorem, and the KKM principle may all be expressed in identical ways, which we prove. On convex spaces in the Lassonde sense, our outcomes manifest as a KKM type theorem, multiple fixed point theorems, and maximum element theorems. In particular, by using our variant of the KKM theorem, we provide an open and obvious demonstration of the Kakutani theorem.

Research methodology

The research methodology for studying the applications of fixed points to solving linear and nonlinear integral equations involves a combination of theoretical analysis, numerical methods, and computational experimentation. The following methodology outlines the steps and procedures involved in investigating these applications:

Results and discussion

The application of fixed-point theorems, such as Banach's Fixed-Point Theorem and Schauder's Fixed-Point Theorem, provides a powerful framework for solving linear and nonlinear integral equations. These results stem from the following theoretical and numerical analyses:

Let $J = [0, T]$ be a bounded interval in $\mathbb{R}$, for some $T \in \mathbb{R}$. Consider the following Voltera-Hammerstein nonlinear integral equation:

$$x(t) = f\big(t, x(t)\big) + \int_0^t k(t,s)g\big(s, x(s)\big)ds$$

for all $t \in J$, where $f : J \times \mathbb{R} \to \mathbb{R}$, $k : J \times J \to \mathbb{R}$ and $g : J \times \mathbb{R} \to \mathbb{R}$ are measurable both in t and s on $\mathbb{R}$.

The class of continuous real-valued functions on J is denoted by C(J, ℝ), and when we say that a function x ∈ C(J, ℝ), we indicate that it is a solution of integral The standard supremum norm || || in C(J, ℝ) is defined as when

$$k \times k = \sup t \in |x(t)|$$

$$k \times k = \sup t \in J|x(t)|.$$

By $L^1(J, \mathbb{R})$ we denote the space of Lebesgue integrable functions on J and the norm || || in $L^1(J, \mathbb{R})$, defined by

$$\|x\|_{L^1} = \int_0^T |x(t)|\, dt$$

Assume the following conditions:

(C_1) $\int_0^T \sup_{0\le s\le t} |k(t,s)|dt = M_1 < \infty$

(C_2) for all $x \in L^1(J,\mathbb{R})$, $g(s,x(s)) \in L^1(J,\mathbb{R})$, there exists $M_2 > 0$ such that,

$$|g(s),x(s) - g(s,y(s))| \le M_2|x(s) - y(s)|, \forall\, x,y \in L^1(J,\mathbb{R}), s \in J,$$

(C_3) for all $x,y \in L^1(J,\mathbb{R})$, there exists $M_3 > 0$ such that

$$|f(t,x(t) - f(t,y(t))| \le M_3\|x - y\|\,,$$

(C_4) defiene $Ax(t) = f\,(t,x(t))$ and

$$Bx(t) = \int_0^t k(t,s)g(s,x(s))ds,$$

with $|ASx(t) - SAx(t)| \le \mathbb{R}\, diam(P\,C(A,S))$, for some $x(t)$ satisfying $Ax(t) = Sx(t), \mathbb{R} > 0$, where $x(t)$ is defined as in (4.79).

Theorem Under the assumption $(C1) - (C4)$ the nonlinear integral equation has a unique solution in L^1(J, R) with

$$\frac{M_3}{1 - M_1M_2} < 1$$

Proof. Define

$$Ax(t) = f(t,x(t)),$$

$$Bx(t) = \int_0^t k(t,s)g(s,x(s))ds,$$

where I is the identity operator on L^1(J, ℝ) and A,B and S are operators from L^1(J, ℝ) into itself. Clearly, Ax, Bx ∈ L^1(J, ℝ). By condition(C_3), we have

$$\|Ax - Ay\| = |f\,(t,x(t)) - f\,(t,y(t))| \le M3\|x - y\| \quad (4.83)$$

$$\|Bx - By\| = \int_0^T |Bx(t) - By(t)|\,dt,$$

$$= \int_0^T \left| \int_0^t k(t,s)g(s,x(s))ds - \int_0^t k(t,s)g(s,y(s))ds \right| dt,$$

$$\le \int_0^T \left[\sup_{0\le s\le t} |k(t,s)|\right] \int_0^t |g(s,x(s)) - g(s,y(s)|ds$$

$$\le M_1M_2 \int_0^T |x(s) - y(s)|\,ds = M_1M_2\|x - y\|_{L^1}$$

$$\|Bx - By\| = \int_0^T |Bx(t) - By(t)|\,dt$$

$$= \int_0^T \left| \int_0^t k(t,s)g(s,x(s))ds - \int_0^t k(t,s)g(s,y(s))ds \right| dt,$$

$$\le \int_0^T \left[\sup_{0\le s\le t} |k(t,s)|dt\right] \int_0^t |g(s,x(s)) - g(s,y(s))|ds$$

$$\le M_1M_2 \int_0^T |x(s) - y(s)|\,ds = M_1M_2\|x - y\|_{L^1}$$

Now, by Equation we have

$$\|Sx - Sy\| = \|(I - B)x - (I - B)y\| = \|(x - y) - (Bx - By)\| \ge \|x - y\| - \|Bx - By\| = (1 - M_1M_2)\|x - y\|$$

From, we get

$$Sx(t) = (I - B)x(t) = x(t) - \int_0^t k(t,s)g\big(s,x(s)\big)ds\,,$$

$$\| Ax - Ay\| \le \frac{M_3}{1 - M_1M_2}\,\|Sx - Sy\|$$

Hence, by Corollary, there exists a unique common fixed point $u \in L^1(J,\mathbb{R})$ such that Au = Su = u, hence, u is the only solution to the equation (4.79). A natural application of Theorem is shown in the following example.

Consider the following integral equation

$$x(t) = \frac{t}{3(1+t^2)}\,x(t) + \int_0^t \sin(2ts)\frac{x(s)}{2+s}ds$$

for all $s, t \in [0,1]$.

Comparing the Equation we get

$$f\big(t,x(t)\big) = \frac{t}{3(1+t^2)}x(t), k(t,s) = \sin(2ts)\,, g\big(s,x(s)\big) = \frac{x(s)}{2+s}, t \in [0,1], x \in \mathbb{R}$$

Clearly,

$$\left|\frac{t}{3(1+t^2)}x(t) - \frac{t}{3(1+t^2)}y(t)\right| \le \frac{1}{3}|\,x(t) - y(t)|\text{ , and}$$

$$|g\big(s,x(s)\big) - g\big(s,y(s)\big)| = \left|\frac{x(s)}{2+s} - \frac{y(s)}{2+s}\right| = \left|\frac{1}{2+s}\big(x(s) - y(s)\big)\right| \le \frac{1}{2}|x(s) - y(s)|$$

Here,

$\sup_{0\le t\le 1}|k(t,s)| = 1 < \infty$, also $\frac{M_3}{1-M_1M_2} = \frac{1/3}{1-1/2} = \frac{2}{3} < 1$

Also, condition(C_4) is satisfied for some $\mathbb{R} > 0$. Thus, all the conditions of satisfied, so it guarantees that there exists a solution of equation in $L^1([0,1],\mathbb{R})$.

When there is a unique common fixed point and a unique point of coincidence among provided single-valued mappings, Doric et al. (2012) demonstrated that weak compatibility may sometimes be reduced to weak compatibility. When there is a unique coincidence point (and a unique common fixed point) among the provided maps, Alghamdi et al. (2012) demonstrated that JH-operators may sometimes be reduced to weakly compatible mappings.

It is noted that PD-operator pairs guarantee the presence of a coincidence point, in contrast to the idea of weak compatibility. Therefore, under relaxed circumstances with applications, PD-operator pairs and D-operator pairs are stronger than weakly compatible mappings. It follows that PD-operator pairs, D-operator pairs, and all the generalizations of commutativity coincide if there are just two maps and a single coincidence point (which happens to be the unique fixed point). When one can demonstrate the outcomes for such maps under other kinds of circumstances or Lipschitz-type conditions, such pairings also have their own practical use.

Conclusion

In conclusion, the application of fixed point theory to the solution of both linear and nonlinear integral equations plays a crucial role in various fields of mathematics and applied sciences. By leveraging the existence and uniqueness results of fixed points, such as those provided by Banach's Fixed Point Theorem, we can derive efficient methods for solving these equations. Fixed point methods provide an elegant framework for both analytical and numerical solutions, especially when dealing with complex nonlinear systems.

For linear integral equations, fixed point approaches help in establishing the existence and uniqueness of solutions, while also offering constructive methods for approximating these solutions. In the case of nonlinear integral equations, fixed point techniques allow for a deeper understanding of the behavior of solutions, and under certain conditions, they can lead to the development of iterative methods that converge to the true solution. The versatility of fixed point theory, coupled with the ability to handle boundary conditions and the nature of integral operators, makes it an indispensable tool for tackling a wide range of problems in mathematical analysis, physics, engineering, and other applied disciplines.

Thus, the use of fixed point methods is not only fundamental in theoretical investigations but also vital in practical computational applications, providing a robust approach for obtaining reliable solutions to complex integral equations.

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