A COMPREHENSIVE OVERVIEW OF DIOPHANTINE EQUATIONS, INCLUDING EARLY SOLUTIONS AND FAMOUS CONJECTURES, TRACING THEIR ORIGINS

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ABSTRACT

Solving the Diophantine equation has fascinated mathematicians from various civilizations. In this paper, we propose the resolution of quadratic Diophantine equations with integer coefficients. Our contribution consists of generalizing certain results which have already been developed in the literature. This paper also proposes the criterion on the solvability of the quadratic Diophantine equation here studied. We consider the Diophantine equation of the title which was re- cently solved, in terms of the number of solutions to it for a 2 {1,2,4}, in (1). However, a counterexample was provided in (6). We provide another coun- terexample and show that both the example herein and the one in (6) are results of Ljunggren (7) from the early 1940s. Given that these are all the omissions from (1), this secures the study of the equation in the title. For relatively prime D1,D2 2 N (the natural numbers), D = D1D2, k 2 N, prime to D, , 2 {1, p 2,2}, with , = 2 if k is even, the Diophantine equation (1) D1x2 + D2 = ,2kn; x,n 2N

Keywords: *Biquadrates; Quartic diophantine equation, Homogeneous Ternary Quadratic, Integral solutions*

INTRODUCTION

An equation in two or more than two unknowns is called an indeterminate equation. Moreover a system of equations is called indeterminate if the number of equations is less than that of the unknowns. The theory of indeterminate equations plays a significant role in the theory of higher Arithmetic and has a marvelous effect on credulous people and always occupy a remarkable position due to its historical importance. Diophantus, one of the Alexanderian mathematician who initiated the study of many indeterminate equations in his Arithmetic, made systematic use of algebraic symbols. He was the first mathematician to make such efforts towards developing of a symbolism for the powers of algebraic expressions. Practically nothing is known of the Diophantus. It is said that he lived in Alexandria sometime around 250 A.D. The only evidence as to the days of his activity is that the Bishop of Laotica, who began his episcopatic in 270 A.D. dedicated a book on Egyptian computation to his friend Diophantus. Diophantus wrote three books, but his main work is his Arithmetica which is memorable. The most ancient manuscript of Arithmetica was written in 13thcentury about a thousand years after the original one appeared. So we are quite uncertain about the symbols used by Diophantus himself and as to the various interpolations that may have been made by medieval copyists.

Diophantine equations is the branch of Number Theory concerned with determining integer solutions of algebraic equations with one or more variables. Diophantus was interested in exact solutions rather then the approximate solutions considered perfectly appropriate. He found interest in polynomial equations in one or more variables for which it is necessary to find a solution to either integers or rational numbers. "Diophantus' youth lasts 1/6 of his life. He grew a beard after 1/12 more of his life. After 1/7 more of his life, Diophantus married. Five years later, he had a son. The son lived exactly half as long as his father and Diophantus died just four years after his son's death. All of this totals the years Diophantus lived."VLet D be the number of years Diophantus lived and let S be the number of years his son lived. Then the above word problem gives the two equations

$$D = \left(\frac{1}{6} + \frac{1}{12} + \frac{1}{7}\right)D + 5 + S + 4$$

$$S = \frac{1}{2}D.$$

LITERATURE REVIEW

Sankar Sitaraman (2003) discussed the unsolvability condition of the equations of the form $x^p + y^p = pcz^p$ which were discussed by him in 2000. He presented the following theorem. For any fixed odd positive integer *n<p-4* and any integer *c* divisible only by primes of the form *kp-1* where (*k*,*p*)=1, assume

(i) At least one of $C_p^{(3)}, C_p^{(5)}, \ldots, C_p^{(n)}$ is non-trivial.

(ii) $C_p^{(i)}=0$ for $p-n-1 \le i \le p-2$.

(iii) $2^i \ne 1 \bmod (p)$ for $1 \le i \le n+1$.

Let q be an odd prime such that $q \equiv 1 \bmod (p)$, and such that there is a prime ideal Q over q in Q(ζ) whose ideal class is of the form $I^p J$ where J is non-trivial, not a *p*th power and $J \in C_p^{(3)} \oplus C_p^{(5)} \oplus \ldots \oplus C_p^{n}$.

Arif, Fadwa and Abu Muriefah (2002) discussed the Diophantine equation . They showed that the equation when q is an odd prime, q is not congruent to 7(mod8), n is an odd integer , n is not a multiple of 3 and (n,h)=1 where his the class number of the field has exactly two families of solutions given by

q=19, n=5, k=5M, x= ,y=55.

q=341, n=5, k=5M, x=2759646. , y= .

They further showed that the equation , when n and q satisfy the above conditions, has no solution for (q,x)=1, except when q=19.341.

Michael A. Bennett & Gary Walsh (1999) discussed the Diophntine equation. They showed that this Diophantine equation has at most one integral solution in x, y if b and d are positive integers and b>1. They also gave an explicit characterization of this solution in terms of fundamental units of associated quadratic field. The Diophantine equation seems to have been discussed first by Gerardin (quoted by Dickson, pp. 647-48) although numerical solutions for the particular cases h=2 and h=5 were discussed by Grigorief and Werebrusow (quoted by Dickson, p. 647)

Ajai Chaudhry (1999) discussed the quartic Diophantine equation where and obtained a necessary and sufficient condition for the existence of non-trivial solution of this equation. He obtained the integer solution of the equation provided an elementary method for obtaining the complete solution of certain homogeneous Diophantine equations of the type where is an integral form of degree k in the variables , I=1, 2, …n. For in three variables, he obtained non-trivial solutions of fourth and fifth degree equations.

The Diophantine Equation

An equation in two or more than two unknowns is called an indeterminate equation. Moreover a system of equations is called indeterminate if the number of equations is less than that of the unknowns. The theory of indeterminate equations plays a significant role in the theory of higher Arithmetic and has a marvelous effect on credulous people and always occupy a remarkable position due to its historical importance. Diophantus, one of the Alexanderian mathematician who initiated the study of many indeterminate equations in his Arithmetic, made systematic use of algebraic symbols.

Linear Diophantine Equations

If a, b and c are real numbers, not both a and b equal to zero than the equation ax+by+c=0 is called a linear equation because the corresponding graph is a straight line in xy-plane. If the above equation is to be solved in integers then it is called a Linear Diophantine Equation (in two variables). The study of Diophantine equations is one of the oldest branch of Mathematics. Undoubtedly, one reason for this is the fact that man has used the positive integers much longer than the other number systems. Diophantine equations also provide a natural vehicle for puzzles and problems of mathematical nature.

Diophantine Equation Related to Octahedron:

The surface area S and the volume V of a tetrahedron of each side x are given by $S = 2\sqrt{3}x^2$ and $V = \frac{\sqrt{2}}{3}x^3$ respectively. We consider the Diophantine equation given by V=S. Substituting the values of V and S, we get $\frac{\sqrt{2}}{3}x^3 = 2\sqrt{3}x^2$ M This gives. $x = 3\sqrt{6}$ Thus the Diophantine equation under consideration has no integral solution. But if we consider the Diophantine equation given by. Then we have x=18. Thus in this case the Diophantine equation under consideration is solvable. Similarly if we consider the Diophantine equation given by. Then we have x=3. Thus in this case also the Diophantine equation under consideration is solvable.

Diophantine Equation Related to Dodecahedron:

The surface area *S* and the volume *V* of a dodecahedron of each side *x* are given by $S = 3\sqrt{25+10\sqrt{5}}x^2$ and $V = \frac{(15+7\sqrt{5})}{4}x^3$ respectively. We consider the Diophantine equation given by *V*=*S*. Substituting the values of *V* and *S*, we get

$$\frac{(15+7\sqrt{5})}{4}x^3 = 3\sqrt{25+10\sqrt{5}}x^2 .$$

This gives $x = 12\frac{\sqrt{25+10\sqrt{5}}}{15+7\sqrt{5}}$. Thus the Diophantine equation under consideration has no integral solution. But if we consider the Diophantine equation given by $3\sqrt{25+10\sqrt{5}}V = (15+7\sqrt{5})S$, then we have *x*=12. Thus the Diophantine equation under consideration is solvable.

METHODOLOGY

The equation of ellipse is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \ldots(3.2)$$

where *a* is the semi-major axis and *b* is the semi-minor axis. We have to find the lattice points, i.e. the positive integral values of *x* and *y* lying inside and on the ellipse (3.2) for different values of *a* and *b*.

Now if $a = 2m$, $b = \frac{2n}{\sqrt{3}}$ (*m* and *n* are positive integers) then *x*=*m* and *y*=*n* are the positive integral solution of the equation

(3.2). The lattice points lying inside and on the given ellipse will be the lattice points lying inside and on the rectangle of sides m and n along x-axis and y-axis respectively with one vertex at the origin.

Now if $a = 3m_1$, $b = \frac{3n_1}{2\sqrt{2}}$ (m_1 and n_1 are positive integers) then $x = m_1$ and $y = n_1$ are the positive integral solution of the equation (3.2). The lattice points lying inside and on the given ellipse will the lattice points lying inside and on the rectangle of sides m_1 and n_1 along x-axis and y-axis respectively with one vertex at the origin.

Lattice Points inside the Cardioid:

The equation of cardioid is given by

$$r = a(1+\cos\theta), \qquad \ldots(3.3)$$

where a is a parameter. We have to find the lattice points lying inside and on the cardioid given by the equation (3.3). The parametric form of the cardioid (3.3) is given by

$$x = r\cos\theta = a(1+\cos\theta)\cos\theta$$

$$y = r\sin\theta = a(1+\cos\theta)\sin\theta$$

RESULTS AND DISCUSSION

(i) Diophantine equation $2xy = n(x+y)$**:**

The given Diophantine equation can be written as

$$\frac{2}{n} = \frac{1}{x} + \frac{1}{y}. \qquad \ldots(1.1)$$

The left hand side of the above equation can be written as

$$\frac{2}{n} = \frac{1}{\frac{(n-1)}{2}+1} + \frac{1}{n\left(\frac{(n-1)}{2}+1\right)}. \qquad \ldots(1.2)$$

Comparing equations (1.1) and (1.2), we get $x = \frac{(n-1)}{2}+1$ and $y = n\left(\frac{(n-1)}{2}+1\right)$.

Now if $n \equiv 1 \pmod 2$ then x and y are positive integers. Thus these are the solutions of the above Diophantine equation. Few solutions are given below:

n	x	y
1	1	1
3	2	6
5	3	15
7	4	28
9	5	45
11	6	66
13	7	91
15	8	120
17	9	153

(ii) Diophantine equation $3xy = n(x+y)$**:**

The given Diophantine equation can be written as (1.3)

$$\frac{3}{n} = \frac{1}{x} + \frac{1}{y}.$$

The left hand side of the above equation can be written as

$$\frac{3}{n} = \frac{1}{\frac{(n-2)}{3}+1} + \frac{1}{n\left(\frac{(n-2)}{3}+1\right)}. \qquad \ldots(1.4)$$

Comparing equations (1.3) and (1.4), we get $x = \frac{(n-2)}{3}+1$ and $y = n\left(\frac{(n-2)}{3}+1\right)$.

Now if $n \equiv 2 \pmod 3$ then x and y are positive integers. Thus these are the solutions of the above Diophantine equation. Few solutions are given below:

n	x	y
5	2	10
8	3	24
11	4	44
14	5	70
17	6	102

20	7	140
23	8	184
26	9	234
29	10	290

(iii) Diophantine equation $3xyz = n(xy + yz + xz)$**:**

The given Diophantine equation can be written as

$$\frac{3}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}. \qquad ...(1.5)$$

The left hand side of the above equation can be written as

$$\frac{3}{n} = \frac{1}{\frac{(n-2)}{3}+1} + \frac{1}{n\left(\frac{(n-2)}{3}+1\right)}$$

$$= \frac{1}{n} + \frac{1}{\frac{(n-1)}{2}+1} + \frac{1}{n\left(\frac{(n-1)}{2}+1\right)}. \qquad ...(1.6)$$

Comparing equations (1.5) and (1.6), we get $x = n$, $y = \frac{(n-1)}{2} + 1$ and $z = n\left(\frac{(n-1)}{2}+1\right)$. Now if $n \equiv 1 (\mathrm{mod}\, 2)$ then *x*, *y* and *z* are positive integers. Thus these are the solutions of the above Diophantine equation. Few solutions are given below:

n	*x*	*y*	*z*
3	3	2	6
5	5	3	15
7	7	4	28
9	9	5	45
11	11	6	66
13	13	7	91
15	15	8	120

CONCLUSION

The Diophantine equations related with the surface area and volume of the cylinder have been discussed in two different cases, i.e. *V*=*C* and *V*=*S* where *V*, *C* and *S* are respectively the volume, the curved surface and the total surface of the cylinder. The Diophantine equations related with the surface area and volume of the cone have been discussed in one case only when *V*=*S*. The Diophantine equations related with the surface area and volume of the rectangular parallelepiped have been discussed in four cases. The Diophantine equations related with the surface area and volume of regular polyhedrons have also been discussed. The Diophantine equations related with the surface area and volume of cuboid have been discussed in two cases. The Diophantine equations related with the surface area and volume of tetrahedron has been discussed in three cases, The Diophantine equations related with the surface area and volume of octahedron have been discussed in three cases. The Diophantine equations related with the surface area and volume of dodecahedron an icosahedron have also been discussed.

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