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ANALYSIS OF PHYSICAL SYSTEMS USING LINEAR AND NON-LINEARDIFFERENTIAL EQUATIONS

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ABSTRACT

The goal of the current study is to advance scientific and engineering knowledge and abilities in order to solve practical challenges. In this article, we will investigate the use of differential equations in the context of engineering and population dynamics, aswell as in the fields of applied mathematics, physics, chemistry, and biology. This is a crucial and effective technique for examining the interaction between different dynamical factors.

Keywords: *Dynamical parameters, physical system and non-linear.*

1. INTRODUCTION

The majority of rules in physics, science, mechanics, and classical mechanics take the form of differential equations and are entirely focused on the pace at which dependent variable differential equations change over time. A differential equation is crucial in the process of creating laws and formulas. A differential equation is an equation that has certain function derivatives. A differential equation is just an equation that has a differential coefficient in it. The study of differential equations is very beneficial. Only two of the many possible forms of differential equations—linear and non-linear differential equations—have been explored. A linear differential equation is one that only uses the first order derivative of an unknown function. It will then be referred to as a non-linear differential equation. As a result, if dydx or y' stands for the first order derivative of the unknown function y= f(x), then dydx=y is linear but dydx=y2 is non-linear. For example, a differential equation is said to be linear if it has no terms suchas y2, (y')3, y.y', siny, logy, or ey.

$\frac{dy}{dx} + 2y=2x$, $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y = 0$

$\frac{d^2y}{dx^2} + y^2\frac{dy}{dx} + y = 0$,$(y')^2 + 2y = x$, $y'' +$ are linear differential equations and logy = 0 are non-linear

differential equations.

A function whose derivatives fulfil an equation is a solution to a differential equation. The highest derivatives that occur in a differential equation are arranged according to their order. When differential coefficients are rational and devoid of fractional power, the degree of the differential equation is the power of the highest order derivatives.

2. APPLICATIONS

Differential equations with constraints imposed at a single location are often used to represent engineering issues that are time-dependent. Here are a few inspiring instances from several branches of engineering, science, and technology, along with where and how they might be applied to population dynamics and engineering problems:

1. A simple pendulum example

2. Energy efficiency

3. Radioactive physics in nuclear decay

4. Coupled L-R and RC-Circuit electrical circuits

5. In thermodynamics, Newton's law of cooling is

6. The dynamics of Newton's second law

7. Particle movement in a changing force field

Some of the inspirational instances are also found in population dynamics, such as therise in population overall and the expansion of bacterial populations in agricultural areas, etc.

3. MOTIVATIONAL EXAMPLES

Assume that the only force operating on a basic pendulum is gravity, which consists of a weight strung on a string. The horizontal force on the weight toward the verticalis exactly proportional to sine angle if is the angle that the pendulum's string makes with a vertical line.

i.e.$\frac{d^2\alpha}{dt^2} \propto \sin\alpha$

If remove proportionality constant ,it has to multiply some constant known as proportionality constant say C0where C0> o therefore,we have

$\frac{d^2\alpha}{dt^2}$= - C_0 Sina

$$\text{hence}\frac{d^2\alpha}{dt^2} + C_0 \sin\alpha = 0 \qquad (1.1)$$

Since the second order differential equation is non-linear, it cannot be solved precisely for the function t.

The differential Equation (1.1) transforms into a linear second order differential equation, however, if the angle of suspension is extremely small, or if the sine of is almost equal to.

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$$\frac{d^2\alpha}{dt^2} + C_0\alpha = 0$$

and this can be solved as follows:

$$\therefore D^2\alpha + C_0\alpha = 0, \text{where } D = \frac{d}{dt}$$
$$\therefore (D^2 + C_0)\,\alpha = 0$$
$$\therefore D^2 + C_0 = 0$$

$$\therefore D = +\sqrt{C_0}\,i \text{ and } D = -\sqrt{C_0}\,i$$
$$\text{Hence, } \alpha(t) = a\cos\sqrt{c_0}\,t + b\sin\sqrt{c_0}\,t$$

which is a required solution of the linear second order differential equation.

4. ENERGY CONSERVATIVE

Consider a particle with mass "m" that is travelling straight. If there is no friction anda force is applied at a location x, then F(x) is conservative, which implies that there exists a function V(x), also known as the potential energy (P.E), such that F(x) =- dVdx. Since we know from lower-level physics that a particle with mass "m" has kinetic energy (K.E) equal to

$$K.E = \frac{1}{2}\, m\, v^2 (v) = \frac{dx}{dt}$$

and the velocity of the particle i.e. the rate of change w.r.t time tBut total energy of the particle of mass 'm' is given by

E = K.E + P.E

$$\therefore E = \frac{1}{2}\, m\, v^2 + V(x)$$
$$\therefore E = \frac{1}{2}\, m \left(\frac{dx}{dt}\right)^2 + V(x)$$
$$\therefore m \left(\frac{dx}{dt}\right)^2 = 2(E - V(x))$$
$$\therefore \left(\frac{dx}{dt}\right)^2 = \frac{2}{m}(E - V(x))(1.2)$$

This is a non-linear differential equation of first order which can be solved byvariables of separation method as we can see below:

$$\therefore \frac{dx}{dt} = \pm\sqrt{\frac{2}{m}}(E - V(x))$$
$$\therefore \frac{dx}{\pm\sqrt{\frac{2}{m}}(E - V(x))} = dt$$

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Now, variables are separated and it canbe integrated. therefore ,

$$\pm \int \frac{dx}{\sqrt{\frac{2}{m}(E - V(x))}} = t - t_0$$

t - $t0$ which is a required solution of non-linear differential equation(1.2).

Note that square root is there, so minus sign will be suitable for constant $t0$.

5. RADIOACTIVE DECAY

Carbon 14 is a radioactive isotope of stable carbon 12. If Q(t) denotes the amount ofcarbon C14 at time t, then Q is known to satisfy the differential equation.

$$\frac{dQ}{dt} = -\lambda Q$$

where λ is

$$\text{Now, } \frac{dQ}{dt} = -\lambda Q$$

$$\therefore \frac{dQ}{Q} = -\lambda dt$$

$$\therefore \log Q = -\lambda t + c',$$

where c' is an integrating constant,

$$\therefore Q = e^{-\lambda t + C'}.$$

Thus, Q= $Ce{-}\lambda t$, where C= ec'.

This shows that the Radioactive Decay decreases exponentially at time t.

6. RC- CIRCUIT

In an RC circuit, the applied e.m.f is a constant E. Given that $dQdt = i$ where Q is thecharge in the capacitor, i is the current in the circuit, R the resistance and C the capacitance the equation for the circuit is

$$Ri + \frac{Q}{C} = E.$$

Given initial charge is zero.

Now, given thatthe equation for the circuit is $Ri + \frac{Q}{C} = E$

$$\therefore R.\frac{dQ}{dt} + \frac{Q}{C} = E$$

$$\therefore \frac{dQ}{dt} + \frac{Q}{CR} = \frac{E}{R}$$

$$\therefore DQ + \frac{Q}{CR} = \frac{E}{R}, \text{ where } D = \frac{d}{dt}$$

$$\therefore \left(D + \frac{1}{CR}\right) Q = \frac{E}{R}$$

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The auxiliary equation is $(D + \frac{1}{CR}) = 0.$ given by

$$D = -\frac{1}{CR}$$

The complimentary function is Q(t) = k $e^{-t}CR$ where k is an arbitrary constant .The particular integral is given by

$$P.I = \frac{1}{D+\frac{1}{CR}}\frac{E}{R} = e^{-t/CR}\int e^{t/CR}.\frac{E}{R}.dt = e^{-t/CR}.\frac{E}{R}\frac{e^{t/CR}}{\frac{1}{CR}} = CE.$$

Therefore, the charge in the capacitor is given byQ(t) = k $e^{-t}CR$ + CE.
But , initial charge is zero i.e. Q(0) = 0.0 = Q(0) = k + CE
∴ k = - CE

Hence, the charge in the capacitor is given byQ(t) = - CE $e^{-t}CR$ + CE.
i.e. Q(t) = CE (1 - $e^{-t/}CR$) which is required charge in an RC- Circuit.

7. NEWTON'S LAW OF COOLING IN THERMODYNAMICS

Suppose an object has temperature T(t) at time t.Newton's law of cooling states that

$$\therefore \frac{dT}{dt} = -k(T - E)$$

$$\therefore \frac{dT}{(T-E)} = -k\,dt$$

T' = - k (T - E) ,where E is the environment temperature and k is the constantNow, T' = - k (T - E)

By integrating,

log(T - E) = - k t + c where c is an integrating constant.

∴ (T - E) = e^{-k} t + c

∴T = E + t0e^{-k} t which is a required temperature.

8. APPLICATION TO POPULATION DYNAMICS

The rate of change of unemployed of the country in certain year t directly proportionalto the number of unemployed people.

$$\text{i.e. } \frac{dU}{dt} \propto U,$$

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where U(t) is the number of unemployed in the country in year t.

If remove proportionality content,it has to multiply some constant therefore , $\frac{dU}{dt}$ $=\lambda U$

Where λ is a proportionality constant and U(t) is the number of unemployed in the $\frac{dU}{dt}$ country.

$-\lambda$ U=0

∴ DU-λ U =0 , where D $=\frac{d}{dt}$

∴ (D-) U =0

So, D=λ

Hence, U(t) = C $e^{\lambda t}$ where C is a constant to be determined.Initially , U(o) = 1 hence C= 1

∴U(t) = $e\lambda t$

This shows that unemployment of the country grows exponentially.

The population of a certain organisms at time t is assumed to satisfy the first order linear differential equation

$$\frac{dP}{dt} = \lambda P(1 - \frac{P}{E}),$$

where P(t) is the number of population at time t,λ and E are positive constants.Given differential equation ((1.3)

$$\frac{dP}{dt} = \lambda P(1 - \frac{P}{E}),$$

$$\therefore \frac{dP}{P(1-\frac{P}{E})} = \lambda\, dt$$

$$\int \frac{dP}{} = \lambda t + t_0$$

where t0 is an integrating constant

Now, consider

$$\frac{1}{P(1-\frac{P}{E})} = \frac{A}{P} + \frac{B}{(1-\frac{P}{E})}$$

∴1 = A $(1 - \frac{P}{E})$ + B.P

when P = E, B = $\frac{1}{E}$ and when P = 0, A= 1.

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$$\therefore \int \left(\frac{1}{P} + \frac{\frac{1}{E}}{(1-\frac{P}{E})}\right) dP = \lambda\, t + t_0$$

$$\therefore \int \left(\frac{1}{P} + \frac{1}{(E-P)}\right) dP = \lambda\, t + t_0$$

$$\therefore \log P - \log (E - P) = \lambda\, t + t_0$$

$$\therefore \log \frac{P}{(E-P)} = \lambda\, t + t_0$$

$$\therefore \frac{P}{(E-P)} = e^{\lambda\, t + t0}$$

$$\therefore P = Ee^{\lambda\, t + t0} - Pe^{\lambda\, t + t0}$$

$$\therefore P + Pe^{\lambda\, t + t0} = Ee^{\lambda\, t + t0}$$

∴ $P(1 + e^{\lambda\, t + t0}) = Ee^{\lambda\, t + t0}$ as required solution of linear differential equation (1.3).

9. GROWTH OF BACTERIA

A certain species of bacteria grows according to

$\frac{dN}{dt} =$ λN with N(0) = $N0$

where N(t) is the amount of bacteria at time t, λ is a positive constant is the growthrate and $N0$ is the initial amount when time t = 0.

Now λN $\frac{dN}{dt} =$

This is a first order linear differential equation and it can be solved by variables of separation method, hence $\frac{DN}{N} = \lambda$t.By integrating on both side, we have log N =λt + c',

therefore, N = $ce^{\lambda t}$ where c = $e^{c'}$.

But $N0$ = N(0) = c $e0$ = c i.e. c = $N0$.

Hence, N = $N0\ e\lambda$t which is the required solution of the linear differentialequation(1.4).

10. CONCLUSION

From the above motivational examples one can see that a linear differential equation and non-linear differential equation has wide range of applications for solving engineering problems,population dynamics as well as other branch of scientific problems of the physical system.So this is an important and powerful tool for analyzing the relationship among various dynamical parameters in engineering field.

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